THE STRUCTURE OF GRAPHS ON n VERTICES WITH THE DEGREE SUM OF ANY TWO NONADJACENT VERTICES EQUAL TO $n-2$

Do Nhu An[a*]

[a]*The Faculty of Information Technology, Nha Trang University, Khanh Hoa, Vietnam*
[*]*Corresponding author: Email: andn@ntu.edu.vn*

Article history
Received: January 10th, 2021
Received in revised form (1st): May 19th, 2021 | Received in revised form (2nd): July 13th, 2021
Accepted: July 16th, 2021
Available online: October 4th, 2021

Abstract

Let G be an undirected simple graph on n vertices with the degree sum of any two non-adjacent vertices equal to $n-2$ and let $\alpha(G)$ be the cardinality of a maximum independent set of G. We show, for $n \geq 3$ is an odd number then $\alpha(G)=2$ and G is a disconnected graph; for $n \geq 4$ is an even number then $2 \leq \alpha(G) \leq (n+2)/2$, where if $\alpha(G)=2$ then G is a disconnected graph, otherwise G is a connected graph.

Keywords: Connected graph; Disconnected graph; Maximum independent set; Regular graph.

DOI: http://dx.doi.org/10.37569/DalatUniversity.11.4.830(2021)

Article type: (peer-reviewed) Full-length research article

1. INTRODUCTION

The concepts and symbols in this article are referenced from the *Handbook of Combinatorics* (Graham et al., 1995). Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices, where $V(G)$ is the vertex set and $E(G)$ is the edge set of graph G. We use $|V(G)|$ and $|E(G)|$ to denote the number of vertices and edges of G. In G, the edge of two vertices u and v is denoted by (u, v), the degree of vertex v is denoted by $deg(v)$, and the minimum degree of the vertices is denoted by δ or $\delta(G)$. A graph on n vertices is called *complete* and denoted by K_n if its vertices have degree $n-1$. A graph is called a k *-regular graph* if all its vertices have degree k. A subset of the vertices in a graph is called *independent set* if no two vertices in this set are adjacent. *A maximum independent set* is an independent set that is not a subset of any other independent set. The cardinality of a maximum independent set in G is denoted by $\alpha(G)$. A subset of the vertices in a graph is called a *clique* if any two of its vertices are adjacent.

The graph $H = (W, F)$ is called a *subgraph* of $G = (V(G), E(G))$ if $W \subseteq V(G)$ and $F \subseteq E(G)$. Let v be a vertex of G; we use $G - v$ to denote the subgraph which is obtained by deleting vertex v and edges attached to v from G. Likewise, if $B \subseteq V(G)$, then $G - B$ is a subgraph of G obtained by deleting B from G. A graph is *connected* if any two of its vertices are connected by a path. A *component* of G is a maximal connected subgraph of G. The number of components of G is denoted by $\omega(G)$.

Now, we use the notation $\sigma_2(G) = n - 2$ to indicate that the graph G on n vertices with the degree sum of any two nonadjacent vertices in G is equal to $n - 2$ and $G(n) := \{G: |V(G)| = n, \sigma_2(G) = n - 2\}$.

An (2008, 2019) has defined the structure of graphs in $G(n) = \{G: |V(G)| = n, \sigma_2(G) = n - 1\}$ and proved that recognizing the Hamiltonian graph in $G(n)$ is an easy problem. In this article, we will define the structure of graphs in $G(n) = \{G: |V(G)| = n, \sigma_2(G) = n - 2\}$ and show for $n \geq 3$ is an odd number and for every $G \in G(n)$ that $\alpha(G) = 2$ and G is a disconnected graph. We also show for $n \geq 4$ is an even number that $2 \leq \alpha(G) \leq (n + 2)/2$ and that G is a disconnected graph if $\alpha(G) = 2$. Otherwise, G is a connected graph.

2. RESULTS

Let $n \geq 3$ and $G \in G(n) = \{G: |V(G)| = n, \sigma_2(G) = n - 2\}$. In G, a vertex of degree $n - 1$ is called a *total vertex*, and the set of total vertices in G is denoted by $T(G)$.

For every $G \in G(n)$, we first note by $\sigma_2(G) = n - 2$ that $G \neq K_n$.

Suppose that u and v are any two nonadjacent vertices in G. We denote the set of vertices that are not adjacent to u by N_u and the set of vertices that are not adjacent to v by N_v. Then $Z := V(G) \backslash N_u \cup N_v$ is a set of vertices that are adjacent to both v and u, and $A := N_u \cap N_v$ is a set of vertices that are not adjacent to v and u. Obviously, $V(G) = Z \cup N_u \cup N_v$ and $T(G) \subseteq Z$.

Remark 1. Let $n \geq 3$ and $G \in G(n)$. Then $|Z| = |A|$.

Proof.

For every $u, v \in V(G)$ and $(u, v) \notin E(G)$, we have $|N(u)| = n - 1 - deg(u)$, $|N(v)| = n - 1 - deg(v)$, and $deg(u) + \deg(v) = \sigma_2(G) = n - 2$. By the inclusion-exclusion principle, $|Z| = |V(G)| - |N_u \cup N_v| = |V(G)| - (|N_u| + |N_v| - |A|) = n - [n - 1 - deg(u) + n - 1 - deg(v) - |A|] = |A|$ and therefore $|Z| = |A|$.

We are interested in two cases of the number of vertices of G.

2.1. The case where n is an odd number

Theorem 1. Let $n \geq 3$ be an odd number and $G \in G(n)$. Then G is a disconnected graph.

Proof.

First, we prove that in G any two nonadjacent vertices have different degrees. (1)

Indeed, let u, v be two nonadjacent vertices in G and $deg(u) = deg(v)$. Then, by $\sigma_2(G) = n - 2$ and $deg(u) + deg(v) = n - 2$, it follows that $deg(u) = deg(v) = (n - 2)/2$, which is a contradiction with n is an odd number. Therefore, $deg(u) \neq deg(v)$.

Next, we will prove that $V(G) = N_u \cup N_v$ and $N_u \cap N_v = \emptyset$. (2)

Without loss of generality, we may assume that $\delta = deg(u) < deg(v) = n - 2 - \delta$, where $0 \leq \delta \leq [(n - 2)/2]$. Since $A = N_u \cap N_v$ is a set of vertices that are both nonadjacent to u and v, it follows that $A = \emptyset$. (If not, let $a \in A$ and by $\sigma_2(G) = n - 2$, $deg(u) + deg(v) = deg(u) + deg(a) = deg(a) + deg(v) = n - 2$. This shows that $deg(u) = deg(v) = deg(a) = (n - 2)/2$, a contradiction with n being an odd number.) By Remark 1 and $A = \emptyset$, we have $Z = \emptyset$ and therefore $V(G) = N_u \cup N_v$, $N_u \cap N_v = \emptyset$.

In addition, by (1) and $\sigma_2(G) = n - 2$, and since vertex $v \in N_u$ is not adjacent to the vertices of N_v in G, it follows that the vertices of N_v have degree δ (similar to the degree of vertex u) and that these vertices are adjacent in G. In other words, the vertices of N_v form a clique $K_{\delta+1}$ in G. Also, the vertices of N_u have degree $n - 2 - \delta$ (similar to the degree of vertex v) and the vertices of N_u form a clique $K_{n-1-\delta}$ in G. And by (2), it follows that G is a disconnected graph and is denoted by $G = K_{\delta+1} \oplus K_{n-1-\delta}$, where $0 \leq \delta \leq [(n - 2)/2]$.

Theorem 1 is proved.

Figure 1 illustrates disconnected graphs corresponding to $\delta = 0, 1, 2$ in $G(7)$.

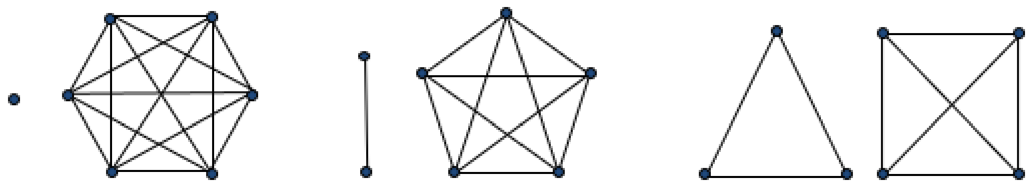

Figure 1. Disconnected graphs in $G(7)$

2.2. The case where n is an even number

Theorem 2. Let $n \geq 4$ be an even number, $G \in G(n)$, and S is an independent set in G. Then

a) if $|S| \geq 3$, the vertices of S have degree $(n-2)/2$ in G.

b) if G is a disconnected graph, G has exactly two components.

Proof.

a) Indeed, let x, y, z be any three nonadjacent vertices of S. Then, by $\sigma_2(G) = n-2$ and $deg(x) + deg(y) = deg(x) + deg(z) = deg(y) + deg(z) = n-2$, it follows that $deg(x) = deg(y) = deg(z) = (n-2)/2$. Moreover, because the vertices x, y, z are chosen arbitrarily, we can say that the vertices of S have degree $(n-2)/2$ in G.

b) Suppose that G has more than two components, and x, y, z are three arbitrary vertices such that each vertex belongs to a component of G. Then, by Theorem 2a, $deg(x) = deg(y) = deg(z) = (n-2)/2$. This shows that each component in G has at least $1 + (n-2)/2$ vertices and that the number of vertices in G is $n = |V(G)| \geq 3(1 + (n-2)/2) = 3n/2 > n$, a contradiction. Therefore, G has only two components.

Theorem 3. Let $n \geq 4$ be an even number and $G \in G(n)$. Then

(a) $0 \leq |T(G)| \leq \delta \leq (n-2)/2$.

(b) $2 \leq \alpha(G) \leq (n+2)/2$.

(c) $\alpha(G) = (n+2)/2 \Leftrightarrow |T(G)| = (n-2)/2$.

(d) $\alpha(G) = n/2 \Rightarrow |T(G)| = 0$.

Proof.

a) Clearly, $\delta \leq (n-2)/2$. Indeed, because if $\delta > (n-2)/2$, then $n-2 = \sigma_2(G) \geq 2\delta > 2(n-2)/2 = n-2$, which is a contradiction. Moreover, each total vertex must be adjacent to other vertices in G, so that the degree of each vertex is not less than $|T(G)|$, i.e., $\delta \geq |T(G)|$. Thus, $0 \leq |T(G)| \leq \delta \leq (n-2)/2$.

Recall that for n is an even number and $n \bmod 4 \neq 0$, then $\delta = (n-2)/2$ is an even number and the total vertex has degree $n-1$, an odd number. Therefore, $k = |T(G)|$ must be an even number.

b) Let S be a maximum independent set in G, $\alpha(G) = |S|$.

First, it is clear that by $\sigma_2(G) = n-2$, $G \neq K_n$, and therefore $\alpha(G) \geq 2$. Next, we prove that $\alpha(G) \leq (n+2)/2$. Suppose otherwise, $\alpha(G) > (n+2)/2$. By $n \geq 4$, $|S| = \alpha(G) > (n+2)/2 \geq (4+2)/2 = 3$. By Theorem 2a, the vertices in S have degree $(n-2)/2$. Moreover, each vertex of S must be adjacent to $(n-2)/2$ vertices of $V(G)\backslash S$ in G. But this cannot happen because the number of vertices of set $V(G)\backslash S$ is $|V(G)\backslash S| = n - |S| < n - (n+2)/2 = (n-2)/2$. This contradiction shows that $\alpha(G) \leq (n+2)/2$.

c) Suppose that $\alpha(G) = (n+2)/2$ and S is a maximum independent set in G. By $n \geq 4, |S| = \alpha(G) = (n+2)/2 \geq (4+2)/2 = 3$, so $|S| \geq 3$, and by Proposition 3a, the vertices of S have degree $(n-2)/2$ in G. Moreover, $|V(G)\backslash S| = n - (n+2)/2 = (n-2)/2$ and each vertex of S must be adjacent to $(n-2)/2$ vertices of $V(G)\backslash S$ in G and by $\sigma_2(G) = n-2$, all the vertices of $V(G)\backslash S$ are total vertices; thus, we get $T(G) = V(G)\backslash S$ and $|T(G)| = (n-2)/2$.

Conversely, suppose that $|T(G)| = (n-2)/2$. We will show that $S := V(G)\backslash T(G)$ is a maximum independent set in G. Obviously, each vertex of S must be adjacent to $(n-2)/2 = |T(G)|$ total vertices in G, and by $\sigma_2(G) = n-2$, the vertices in S have degree $\delta = (n-2)/2$ and are nonadjacent in G. Therefore, S is an independent set in G. But $|S| = |V(G)| - |T(G)| = n - (n-2)/2 = (n+2)/2$, and by Theorem 3b, S is a maximum independent set in G, $\alpha(G) = |V(G)\backslash T(G)| = (n+2)/2$.

d) Suppose that S is a maximum independent set of G and $|S| = \alpha(G) = n/2$. We prove that $T(G) = \emptyset$ and so G is a $(n-2)/2$-regular graph.

First, for $n = 4$ it is easy to show by $\alpha(G) = 2$ that $G = K_2 \oplus K_2$ is a 1-regular disconnected graph. Now, we consider the case $n \geq 6$. Let $X := V(G)\backslash S$. By $n \geq 6$, $|S| = |X| = n/2 \geq 3$. By Theorem 2a, the vertices of S have degree $\delta = (n-2)/2$, and therefore the vertices of S must be adjacent to $(n-2)/2$ vertices of X in G. Thus, for each vertex $s \in S$, there exists only one vertex $x \in X$ such that x and s are nonadjacent, and x must be adjacent to some other vertices of S in G. (If not, $S \cup \{x\}$ is an independent set in G, a contradiction for S is a maximum independent set of G.) Moreover, by $deg(x) = (n-2)/2$, there exists a vertex $y \in X$ that is not adjacent to vertex x in G. This shows that X does not contain the total vertex and that G is a $(n-2)/2$-regular graph.

Theorem 3 is proved. □

Theorem 4. Let $n \geq 6$ be an even number and $G \in G(n)$. Then

a) $\alpha(G) = 2$ if and only if G is a disconnected graph.

b) if $3 \leq \alpha(G) \leq (n+2)/2$, G is a connected graph, and G contains k total vertices and $n-k$ vertices of degree $\delta = (n-2)/2$, where $0 \leq k = |T(G)| \leq (n-2)/2$.

Proof.

Suppose that $\alpha(G) = 2$, we will prove that G is a disconnected graph.

Without loss of generality, we can suppose that $S = \{u, v\}$ is a maximum independent set of G, $deg(u) = \delta$, and $deg(v) = n - 2 - \delta$. First, we have $A = \emptyset$ (because if $A \neq \emptyset$ and let $a \in A$, then $\{u, v, a\}$ is an independent set in G, which is a contradiction with $\alpha(G) = 2$). Since $A = \emptyset$ and by Remark 1, $Z = \emptyset$ and $T = \emptyset$, and so we get $V(G) = N_u \cup N_v$, $N_u \cap N_v = \emptyset$. Next, by $\alpha(G) = 2$, each pair of vertices of N_u must be adjacent in G. (If not, let $x, y \in N_u$ and $(x, y) \notin E(G)$, then $\{x, y, u\}$ is an independent set in G, which is a contradiction with $\alpha(G) = 2$.) Therefore, the vertices of N_u form a clique $K_{deg(v)+1} = K_{n-1-\delta}$ in G. Similarly, each pair of vertices of N_v must be adjacent in G, and these vertices form a clique $K_{deg(u)+1} = K_{\delta+1}$ in G. In addition, by $\sigma_2(G) = n - 2$, the vertices of N_v are not adjacent to the vertices of N_u. These results show that G is a disconnected graph and $G = K_{\delta+1} \oplus K_{n-1-\delta}$, $0 \leq \delta \leq (n-2)/2$.

Conversely, let G be a disconnected graph. We will show that $\alpha(G) = 2$.

By Theorem 2b, graph G has two components. Let $G = G_1 \oplus G_2$, where G_1 and G_2 are connected subgraphs of G. We will prove that G_1 and G_2 are complete graphs and so $\alpha(G) = 2$. Therefore, Theorem 4a is true.

Indeed, without loss of generality we may assume that $x \in V(G_1)$, $y \in V(G_2)$, and $deg(x) \leq deg(y)$. Since the vertices in G_1 are not adjacent to the vertices in G_2, and by $\sigma_2(G) = n - 2$, the vertices in G_1 must have the same degree as vertex x and the vertices in G_2 must have the same degree as vertex y. Now we consider the following two cases: $deg(x) = deg(y)$ and $deg(x) < deg(y)$.

For $deg(x) = deg(y)$ and by $deg(x) + deg(y) = n - 2$, we have $deg(x) = deg(y) = (n-2)/2$. Then $|V(G_1)| = (n-2)/2 + 1 = n/2 = |V(G_2)|$. It follows that G_1 and G_2 are complete graphs $K_{n/2}$ and so $G = K_{n/2} \oplus K_{n/2}$. For $deg(x) < deg(y)$ and since $\sigma_2(G) = n - 2$, each pair of vertices in G_1 must be adjacent. In other words, G_1 is a complete graph $K_{deg(x)+1}$. Analogously, each pair of vertices in G_2 must be adjacent and G_2 is a complete graph $K_{deg(y)+1}$, therefore $G = K_{deg(x)+1} \oplus K_{deg(y)+1}$. In both cases above we get the result that G_1 and G_2 are complete graphs.

Note that Theorem 4a is also true for $n = 4$.

b) Let S be a maximum independent set of G. First, since $|S| = \alpha(G) \geq 3$ and by Theorem 2a, all vertices of S have degree $(n-2)/2$ in G. Moreover, by $\sigma_2(G) = n-2$, it follows that each vertex of $V(G)\backslash S$ has degree either $(n-1)$ or $(n-2)/2$ in G. In other words, G contains k total vertices and $(n-k)$ vertices of degree $\delta = (n-2)/2$, where $0 \leq k = |T(G)| \leq (n-2)/2$ (by Theorem 3a).

Next, in order to show that G is a connected graph, we consider the following two cases: $T(G) \neq \emptyset$ and $T(G) = \emptyset$.

- For $T(G) \neq \emptyset$. Clearly, G is a connected graph because G contains the total vertex.
- For $T(G) = \emptyset$. Then, G is an $\delta-$regular graph for $\delta = (n-2)/2$.

Now, suppose otherwise – that G is a disconnected graph. Then, by Theorem 2b, $G = G_1 \oplus G_2$, where G_1 and G_2 are components of G. Without loss of generality, we may assume that $|V(G_1)| \leq n/2 \leq |V(G_2)|$. However, the vertices in G_1 have degree $\delta = (n-2)/2$, so $|V(G_1)| \geq \delta + 1 = (n-2)/2 + 1 = n/2$. It follows that $|V(G_1)| = |V(G_2)| = n/2$. Moreover, by $\sigma_2(G) = n-2$, G_1 and G_2 must be a complete graph $K_{\delta+1}$, and so $G = K_{\delta+1} \oplus K_{\delta+1}$. It follows that $\alpha(G) = 2$, which is a contradiction with the supposition $\alpha(G) \geq 3$. Therefore, G is a connected graph. Theorem 4b is proved.

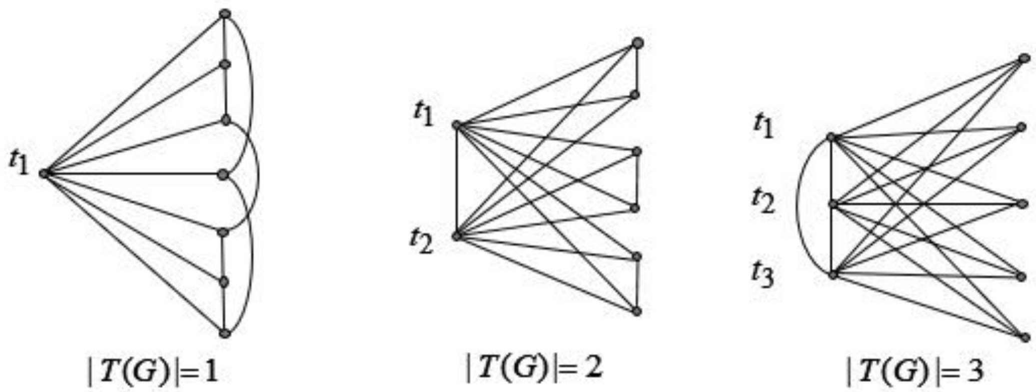

Figure 2. Connected graphs for $|T(G)| = 1, 2, 3$ in $G(8)$

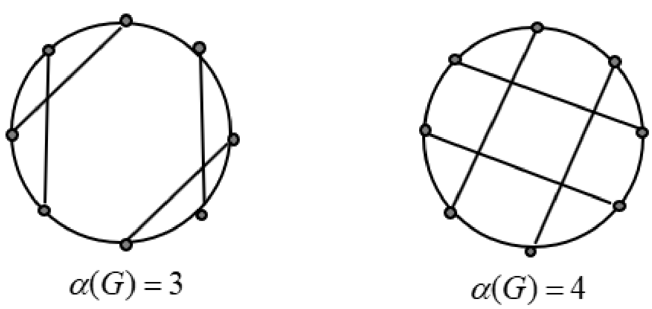

Figure 3. 3-regular graphs in $G(8)$

Figure 2 illustrates connected graphs for $\delta = 3$ and $|T(G)| = 1,2,3$ in $G(8)$, respectively. Figure 3 illustrates 3-regular graphs with $\alpha(G) = 3$ and $\alpha(G) = 4$ in $G(8)$.

Theorem 4 is proved.

3. CONCLUSION

For $G(n) = \{G: |V(G)| = n, \sigma_2(G) = n - 2\}$ and $G \in G(n)$, we have shown that if $n \geq 3$ is an odd number, then G is a family of disconnected graphs $K_{\delta+1} \oplus K_{n-1-\delta}$, $\delta = 0, 1, 2, \ldots, [(n-2)/2]$. For $n \geq 4$ is an even number, there are two cases: If $\alpha(G) = 2$, then G is a family of disconnected graphs $K_{\delta+1} \oplus K_{n-1-\delta}$, $\delta = 0,1,2, \ldots, (n-2)/2$. If $3 \leq \alpha(G) \leq (n+2)/2$, then G is a family of connected graphs that contains k total vertices and $n-k$ vertices of degree $\delta = (n-2)/2$, where $0 \leq k \leq (n-2)/2$. When $k = 0$, G is a δ −regular graph.

REFERENCES

An, D. N. (2008). Some problems about Hamiltonian cycle in special graphs. *The 2nd International Conference on Theories and Applications of Computer Science (ICTACS'09) - Journal of Science and Technology, Vietnam Academy of Science and Technology*, *6*(5A-Special issue), 57-66.

An, D. N. (2019). Recognizing the Hamiltonian graph on n vertices with $\sigma_2(G) = n - 1$ is an easy problem. *International Journal of Advanced Research in Computer Science*, *10*(2), 42-45.

Graham, R. L., Grötschel, M., & Lovász, L. (1995). *Handbook of combinatorics* (1st Ed.). North Holland.