

GEOMETRY AND TOPOLOGY OF EXTERNAL AND SYMMETRIC PRODUCTS OF VARIETIES

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Abstract

We give a brief overview of recent developments on the calculation of generating series for invariants of external products of suitable coefficients (e.g., constructible or coherent sheaves, or mixed Hodge modules) on complex quasi-projective varieties.

Keywords: Characters of representations; External and symmetric products; Generating series; Schur functor; Symmetric groups; Symmetric monoidal category.

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1. INTRODUCTION

This is an expository note intended to give a brief description of some of the main results of our paper Maxim and Schürmann (2020) (see also Maxim & Schürmann, 2018) concerning invariants of external products of suitable coefficients on possibly singular complex quasi-projective varieties. Along the way, we provide a historical overview of the study of geometry and topology of various spaces of objects associated with a given variety.

1.1. Motivation

Recall that to a possibly singular complex quasi-projective variety X , one can associate the following “moduli” spaces of objects:

(a) the n -th external product X^n , which is the product of n copies of X ; it comes equipped with the natural action of the symmetric group Σ_n on n elements.

(b) the n -th symmetric product $S^n X := X^n / \Sigma_n$, which parametrizes effective zero-cycles on X , with the natural projection map denoted by $\pi_n : X^n \rightarrow S^n X$.

(c) the configuration space of ordered n -tuples of distinct points in X , that is,

$$F^n X := \{(x_1, x_2, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\},$$

which is endowed with (the restriction of) the natural Σ_n -action (on X^n).

(d) the configuration space $C^n X := F^n X / \Sigma_n$ of unordered n -tuples of distinct points in X .

(e) the Hilbert scheme $\text{Hilb}^n X$, which is the moduli space for zero-dimensional subschemes of X of length n , describing collections of n (not necessarily distinct) points on X . It comes equipped with a natural morphism $\text{Hilb}^n X \rightarrow S^n X$ to the n -th symmetric product of X , the *Hilbert-Chow morphism*, which takes a zero-dimensional scheme to its associated zero-cycle.

Such moduli spaces of objects associated with a given variety X carry interesting and surprising structures that contain valuable information about the variety X itself. For instance, the symmetric products $S^n X$ were used by Macdonald (1962b) to study the Jacobian variety of X in the case that X is a smooth curve. If X is a smooth complex surface, Cheah (1996) and Göttsche and Soergel (1993) used its symmetric products to understand the topology of Hilbert schemes of points on X . Moreover, the Bridgeland-King-Reid correspondence of Bridgeland et al. (2001) shows that in this case the n -th Hilbert scheme $\text{Hilb}^n X$ can be understood via a Σ_n -equivariant study of X^n .

Configuration spaces, widely used in robot motion planning, appear also in the study of the topology of moduli spaces of stable curves, e.g., in the work of Getzler, Looijenga, etc. Symmetric products and configuration spaces have also been recently

studied from a probabilistic perspective in Vakil and Wood (2015) in relation to stability/stabilization phenomena (see also Kupers & Miller, 2016).

It is therefore important to understand the geometry and topology of such moduli spaces (denoted generically by $M^n X$) built out of a given variety X . The standard approach to compute invariants $\mathcal{I}(-)$ of any of these moduli spaces is to collect them in a *generating series* $\sum_{n \geq 0} \mathcal{I}(M^n X) \cdot t^n$ and to calculate this expression solely in terms of the invariants of X . Then, the corresponding invariant of $M^n X$ is equal to the coefficient of t^n in the resulting expression in terms of invariants of X .

1.2. Historical overview

Many of the generating series formulae in the literature were initially conjectured in (or inspired from) physics and deal mainly with invariants of the moduli space itself.

For instance, in the context of symmetric products, there is a well-known formula due to Macdonald (1962a) for the generating series of the *Betti numbers*

$$b_k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C}),$$

Poincaré polynomial

$$P(X) := \sum_{k \geq 0} b_k(X) \cdot (-z)^k,$$

and topological *Euler characteristic* $\chi(X) = P(X)(1)$ of a compact triangulated space X :

$$\sum_{n \geq 0} P(S^n X)(z) \cdot t^n = \exp \left(\sum_{r \geq 1} P(X)(z^r) \cdot \frac{t^r}{r} \right) \quad (1)$$

and

$$\sum_{n \geq 0} \chi(S^n X) \cdot t^n = \exp \left(\sum_{r \geq 1} \chi(X) \cdot \frac{t^r}{r} \right) = (1-t)^{-\chi(X)}. \quad (2)$$

Macdonald's formula was extended to the Chern-MacPherson classes (see MacPherson, 1974) of symmetric products by Ohmoto (2008). Moonen (1978) obtained the generating series for the *arithmetic genus*

$$\chi_a(X) := \sum_{k \geq 0} (-1)^k \cdot \dim_{\mathbb{C}} H^k(X, \mathcal{O}_X)$$

of symmetric products of a complex projective variety:

$$\sum_{n \geq 0} \chi_a(S^n X) \cdot t^n = \exp \left(\sum_{r \geq 1} \chi_a(X) \cdot \frac{t^r}{r} \right) = (1-t)^{-\chi_a(X)} \quad (3)$$

and, more generally, for the Baum-Fulton-MacPherson homology Todd classes (see Baum et al., 1975) of symmetric products of complex projective varieties. Zagier (1972) proved

such generating series for the *signature* $\sigma(X)$ and *L-classes*, respectively, of symmetric products of compact triangulated (rational homology) manifolds. For instance, Zagier showed that for a complex projective homology manifold X of pure even complex dimension, one has

$$\sum_{n \geq 0} \sigma(S^n X) \cdot t^n = \frac{(1+t)^{\frac{\sigma(X)-\chi(X)}{2}}}{(1-t)^{\frac{\sigma(X)+\chi(X)}{2}}}. \quad (4)$$

If X is a complex quasi-projective variety and

$$h_{(c)}^{p,q,k}(X) := h^{p,q}(H_{(c)}^k(X, \mathbb{Q})) := \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W H_{(c)}^k(X, \mathbb{C})$$

are the *Hodge numbers* of Deligne's mixed Hodge structure on the (compactly supported) cohomology $H_{(c)}^*(X, \mathbb{Q})$, with generating *mixed Hodge polynomial*

$$h_{(c)}(X)(y, x, z) := \sum_{p,q,k \geq 0} h_{(c)}^{p,q,k}(X) \cdot y^p x^q (-z)^k,$$

then the generating series for the Hodge numbers of symmetric products of X is given by Cheah (1996) as

$$\sum_{n \geq 0} h_{(c)}(S^n X)(y, x, z) \cdot t^n = \exp \left(\sum_{r \geq 1} h_{(c)}(X)(y^r, x^r, z^r) \cdot \frac{t^r}{r} \right). \quad (5)$$

Note that for a quasi-projective variety X one has

$$b_k(X) = \sum_{p,q} h^{p,q,k}(X),$$

so one gets back formula (1) by substituting $(y, x) = (1, 1)$ in (5). Similarly, using the relation

$$\chi_{-y}(X) = h(X)(y, 1, 1)$$

for a complex projective variety X , one gets for the *Hirzebruch χ_y -polynomial* the identity:

$$\sum_{n \geq 0} \chi_{-y}(S^n X) \cdot t^n = \exp \left(\sum_{r \geq 1} \chi_{-y^r}(X) \cdot \frac{t^r}{r} \right). \quad (6)$$

Let us also note that for a complex projective manifold X , one also has

$$\chi_a(X) = \chi_0(X) = h(X)(0, 1, 1) \quad \text{and} \quad \sigma(X) = \chi_1(X) = h(X)(-1, 1, 1),$$

so that one gets back (3) and (4) for this case by letting $(y, x, z) = (0, 1, 1)$ and $(y, x, z) = (-1, 1, 1)$, respectively. Generating series for the Hirzebruch χ_y -genus of symmetric products of *smooth* compact varieties, and, more generally, for elliptic genera, were also obtained in Borisov and Libgober (2002).

Hilbert schemes of points on a smooth surface are smooth and their topology is fairly well understood: there exist generating series for their Betti numbers (see Göttsche,

1990), Hodge numbers (see Göttsche & Soergel, 1993), elliptic genus (see Borisov & Libgober, 2003), Grothendieck motives (see Göttsche, 2001), cobordism classes (see Ellingsrud et al., 2001), etc. Much less is known about the Hilbert schemes of points on a smooth variety X of dimension $d \geq 3$ (These are usually not smooth). Cheah (1996) found a generating function expressing the Hodge-Deligne polynomials $h_c(-)(y, x, 1)$ of Hilbert schemes $\text{Hilb}^n X$ in terms of the Hodge-Deligne polynomial of X and those of the punctual Hilbert schemes $\text{Hilb}_{\mathbb{C}^d, 0}^i$ of all zero-dimensional subschemes in the affine space \mathbb{C}^d that are supported at the origin. Cheah’s result was refined in Gusein-Zade et al. (2006), where the notion of *power structure* over the Grothendieck ring of varieties was used to express the generating series of Grothendieck motives of Hilbert schemes of points on a quasi-projective manifold of dimension d as an exponent of that for the affine space \mathbb{C}^d .

It is also natural to investigate such generating series for more “singular” invariants, e.g., for the signature, Hodge numbers, etc., defined via Goresky-MacPherson’s intersection cohomology theory (see Goresky & MacPherson, 1980, 1983) or, more generally, for invariants associated with arbitrary “coefficients.”

In the case of symmetric products, this approach was initiated in Maxim et al. (2011) where, starting with a complex of mixed Hodge modules \mathcal{M} on a quasi-projective variety X , one defines *symmetric power complexes*

$$S^n \mathcal{M} := (\pi_{n*} \mathcal{M}^{\boxtimes n})^{\Sigma_n}$$

on the n -th symmetric product $S^n X$ and derives generating series for their Hodge numbers

$$h_{(c)}^{p,q,k}(S^n X, S^n \mathcal{M}) := h^{p,q}(H_{(c)}^k(S^n X; S^n \mathcal{M})) := \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W H_{(c)}^k(S^n X; S^n \mathcal{M})_{\mathbb{C}}$$

($k \in \mathbb{Z}$), expressed only in terms of the Hodge numbers $h_{(c)}^{p,q,k}(X, \mathcal{M})$ of $H_{(c)}^*(X; \mathcal{M})$. For various choices of coefficients \mathcal{M} on X , the resulting formula recovers, or extends to the singular setting, many of the above-mentioned results about invariants of symmetric products. For example, Cheah’s computation of the Hodge numbers of symmetric products of X corresponds to the choice of the constant Hodge module $\mathcal{M} = \mathbb{Q}_X^H$. Moreover, by considering $\mathcal{M} = IC_X^H$, the Deligne intersection cohomology complex (see Goresky & MacPherson, 1983), Zagier’s formula for the signature of symmetric products of rational homology manifolds is extended to the case of Goresky-MacPherson signatures of symmetric products. A more conceptual proof of these results was given in Maxim and Schürmann (2012) by relating symmetric group actions on external products to the theory of *lambda rings*. In relation to the configuration spaces $C^n X$ in Maxim and Schürmann (2012), one also defines *alternating power complexes* $C^n \mathcal{M}$ of mixed Hodge modules and computes generating series for their corresponding Hodge numbers.

By building on Maxim et al. (2011) and Cappell et al. (2012), Cappell et al. (2017) derive generating series for the Brasselet-Schürmann-Yokura homology *Hirzebruch classes* $T_{y_*}(S^n \mathcal{M})$ (Brasselet et al., 2010) of symmetric powers of a complex of

mixed Hodge modules \mathcal{M} on a variety X . In the special case $\mathcal{M} = \mathbb{Q}_X^H$, this yields a generating series for the *motivic Hirzebruch classes* $T_{y_*}(S^n X)$ of symmetric products of X , recovering as a corollary Ohmoto's generating series formula (Ohmoto, 2008) for the rationalized Chern-MacPherson classes of symmetric products, while at the same time providing a characteristic class version of formula (6).

Furthermore, the symmetric product formula from Cappell et al. (2017) was applied in Cappell et al. (2013) to the study of Hirzebruch-type invariants of Hilbert schemes of points on a quasi-projective manifold and, in the case of a Calabi-Yau 3-fold, it yields a Chern class version of *the Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) conjecture* (Maulik et al., 2006) from Donaldson-Thomas theory. This conjecture, which by now admits several different proofs, predicted a 3-dimensional analogue of Göttsche's generating series formula (Göttsche, 1990) for the Euler characteristics of Hilbert schemes of points on smooth projective surfaces.

Some of the above-mentioned results have been refined in two papers (Maxim & Schürmann, 2018, 2020) mentioned earlier via the equivariant study of external products of varieties and coefficients. This approach is inspired by the BKR correspondence of Bridgeland et al. (2001).

In Maxim and Schürmann (2018), we obtained generating series formulae for *equivariant characteristic classes*, introduced in Cappell et al. (2012), of external and symmetric products of singular complex quasi-projective varieties. More concretely, we considered equivariant versions of homology Todd, Chern, and Hirzebruch classes with values in the *delocalized Borel-Moore homology* of external and symmetric products. As a byproduct, we obtained new equivariant generalizations of the above-mentioned characteristic class formulae for symmetric powers of coefficients from Cappell et al. (2017), particularly in the context of twisting by representations of the symmetric group (e.g., for various other Schur functors). Furthermore, the equivariant techniques developed in Maxim and Schürmann (2018) also yield new (or recover old) generating series identities in the framework of *orbifold cohomology* and *K-theory*, respectively.

In Maxim and Schürmann (2020), we proved generating series formulae for the *characters of virtual cohomology representations of external products* of suitable coefficients (e.g., constructible or coherent sheaves, or mixed Hodge modules) on complex quasi-projective varieties. These formulae generalize some of our previous results from Maxim and Schürmann (2012) and Maxim et al. (2011) for symmetric and alternating powers of such coefficients, and also apply to other Schur functors.

1.3. Goal of this paper

In order to keep the exposition as simple as possible, in this note we give a brief description of the main results and key ideas from Maxim and Schürmann (2020) concerning Poincaré-type identities for invariants of external products of suitable coefficients on complex quasi-projective varieties. While their characteristic class counterparts from

Maxim and Schürmann (2018) are similar in spirit, the presentation of such characteristic class formulae would be much more technically demanding.

2. RESULTS

2.1. Coefficients and their associated invariants

Let X be a complex quasi-projective variety. We consider coefficients $\mathcal{M} \in A(X)$, where $A(X)$ is one of the following:

- (1) $D_c^b(X)$, the bounded derived category of constructible sheaf complexes of \mathbb{C} -vector spaces (e.g., $\mathcal{M} = \mathbb{C}_X, IC_X$);
- (2) $D_{coh}^b(X)$, the bounded derived category of complexes of \mathcal{O}_X -modules with coherent cohomology, if X is projective (e.g., $\mathcal{M} = \mathcal{O}_X$); and
- (3) $D^b\text{MHM}(X)$, the bounded derived category of algebraic mixed Hodge modules on X (e.g., $\mathcal{M} = \mathbb{Q}_X^H, IC_X^H$).

For any $\mathcal{M} \in A(X)$ as above, each (compactly supported) hypercohomology group $H_{(c)}^k(X, \mathcal{M})$ is a finite-dimensional \mathbb{C} -vector space, or a \mathbb{Q} -mixed Hodge structure if $A(X) = D^b\text{MHM}(X)$. Let

$$b_{(c)}^k(X, \mathcal{M}) := \dim H_{(c)}^k(X, \mathcal{M})$$

be the k -th Betti number of \mathcal{M} , with generating *Poincaré polynomial*

$$P_{(c)}(X, \mathcal{M})(z) := \sum_k b_{(c)}^k(X, \mathcal{M}) \cdot (-z)^k \in \mathbb{Z}[z^{\pm 1}].$$

For $\mathcal{M} \in D^b\text{MHM}(X)$, we also let

$$h_{(c)}^{p,q,k}(X, \mathcal{M}) := h^{p,q}(H_{(c)}^k(X, \mathcal{M})) := \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W H_{(c)}^k(X, \mathcal{M})_{\mathbb{C}}$$

be the *Hodge numbers* of (X, \mathcal{M}) , with generating *mixed Hodge polynomial*

$$h_{(c)}(X, \mathcal{M})(y, x, z) := \sum_{p,q,k} h_{(c)}^{p,q,k}(X, \mathcal{M}) \cdot y^p x^q (-z)^k \in \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}].$$

Here, $H_{(c)}^k(X, \mathcal{M})_{\mathbb{C}}$ denotes the underlying \mathbb{C} -vector space of the \mathbb{Q} -mixed Hodge structure on $H_{(c)}^k(X, \mathcal{M})$.

For simplicity of exposition, in what follows we *focus on Poincaré-type identities*, but similar results hold in the mixed Hodge context, for the graded parts with respect to the Hodge and weight filtrations, if $A(X) = D^b\text{MHM}(X)$.

2.2. Cohomology representations of external products

Let X be a complex quasi-projective variety, and consider coefficient theories $A(X)$ as discussed in Section 2.1.

For any $\mathcal{M} \in A(X)$, the external product $\mathcal{M}^{\boxtimes n} \in A(X^n)$ has a natural Σ_n -action (e.g., see Maxim et al., 2011), and there is a Σ_n -equivariant Künneth isomorphism of \mathbb{C} -vector spaces, or \mathbb{Q} -mixed Hodge structures if $A(X) = D^b\text{MHM}(X)$:

$$H_{(c)}^*(X^n, \mathcal{M}^{\boxtimes n}) \simeq H_{(c)}^*(X, \mathcal{M})^{\otimes n}. \quad (7)$$

More generally, if V is a complex Σ_n -representation and $\mathcal{M} \in A(X)$, one can consider *twisted coefficients* $V \otimes \mathcal{M}^{\boxtimes n} \in A(X^n)$ on X^n , with the induced diagonal Σ_n -action. Then the following Σ_n -equivariant Künneth isomorphism of \mathbb{C} -vector spaces, or \mathbb{Q} -mixed Hodge structures if $A(X) = D^b\text{MHM}(X)$ holds:

$$H_{(c)}^*(X^n, V \otimes \mathcal{M}^{\boxtimes n}) \simeq V \otimes H_{(c)}^*(X, \mathcal{M})^{\otimes n}. \quad (8)$$

Here, in the mixed Hodge context, V is regarded as a pure Hodge structure of type $(0,0)$.

In particular, by associating to a representation its character $\text{char}(-)$, we can view the (hyper)cohomology groups of $\mathcal{M}^{\boxtimes n}$ and $V \otimes \mathcal{M}^{\boxtimes n}$ as Σ_n -characters, via

$$H_{(c)}^k(X^n, \mathcal{M}^{\boxtimes n}), H_{(c)}^k(X^n, V \otimes \mathcal{M}^{\boxtimes n}) \in \text{Rep}_{\mathbb{C}}(\Sigma_n) \xrightarrow{\text{char}} C(\Sigma_n),$$

with $\text{Rep}_{\mathbb{C}}(\Sigma_n)$ the Grothendieck group of finite-dimensional complex Σ_n -representations and $C(\Sigma_n)$ the free abelian group of \mathbb{Z} -valued class functions on Σ_n . For a fixed n , we define the generating Poincaré polynomial of these characters by

$$\text{char}(H_{(c)}^*(X^n, V \otimes \mathcal{M}^{\boxtimes n})) := \sum_k \text{char}(H_{(c)}^k(X^n, V \otimes \mathcal{M}^{\boxtimes n})) \cdot (-z)^k \in C(\Sigma_n) \otimes \mathbb{Z}[z^{\pm 1}]. \quad (9)$$

The main question addressed in Maxim and Schürmann (2020) deals with the following.

Problem 2.1. *Compute the Poincaré polynomial $\text{char}(H_{(c)}^*(X^n, V \otimes \mathcal{M}^{\boxtimes n}))$ in terms of invariants of (X, \mathcal{M}) and V , for $\mathcal{M} \in A(X)$ and $V \in \text{Rep}_{\mathbb{C}}(\Sigma_n)$.*

2.3. Approach and main results

Our approach (Maxim & Schürmann, 2020) for solving Problem 2.1 consists of the following key steps:

- (1) compute the generating series $\sum_{n \geq 0} \text{char}(H_{(c)}^*(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n$,

- (2) identify the coefficient of t^n , and
- (3) twist by the representation V and use the multiplicativity of characters.

After composing with the *Frobenius character homomorphism* (see Macdonald, 1979, Ch. 1, Sect. 7):

$$\text{ch}_F : C(\Sigma) \otimes \mathbb{Q} := \bigoplus_n C(\Sigma_n) \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}[p_i, i \geq 1]$$

to the graded ring of \mathbb{Q} -valued symmetric functions in infinitely many variables x_m ($m \in \mathbb{N}$), with $p_i := \sum_m x_m^i$ the i -th power sum function, the generating series of Poincaré polynomials of characters, that is, $\sum_{n \geq 0} \text{char}(H_{(c)}^*(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n$, can be regarded as an element in the \mathbb{Q} -algebra $\mathbb{Q}[p_i, i \geq 1; z^{\pm 1}][[t]]$. The main results of Maxim and Schürmann (2020) can now be stated as follows:

Theorem 2.2. (a) For $\mathcal{M} \in A(X)$, the following generating series identity holds in the \mathbb{Q} -algebra $\mathbb{Q}[p_i, i \geq 1, z^{\pm 1}][[t]]$:

$$\sum_{n \geq 0} \text{char}(H_{(c)}^*(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n = \exp \left(\sum_{r \geq 1} p_r \cdot P_{(c)}(X, \mathcal{M})(z^r) \cdot \frac{t^r}{r} \right).$$

(b) For $V \in \text{Rep}_{\mathbb{C}}(\Sigma_n)$ and $\mathcal{M} \in A(X)$, the following identity holds in the \mathbb{Q} -algebra $\mathbb{Q}[p_i, i \geq 1, z^{\pm 1}]$:

$$\text{char}(H_{(c)}^*(X^n, V \otimes \mathcal{M}^{\boxtimes n})) = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \cdot \prod_{r \geq 1} (P_{(c)}(X; \mathcal{M})(z^r))^{k_r},$$

where, for a partition $\lambda = (k_1, k_2, \dots)$ of n (i.e., $\sum_{r \geq 1} r \cdot k_r = n$) corresponding to a conjugacy class of an element $\sigma \in \Sigma_n$, we set: $z_\lambda := \prod_{r \geq 1} r^{k_r} \cdot k_r!$, $\chi_\lambda(V) := \text{trace}_\sigma(V)$, and $p_\lambda := \prod_{r \geq 1} p_r^{k_r}$.

2.4. Applications

Various specializations of Theorem 2.2 (and of its Hodge-theoretic analogue) can be obtained by making special choices of the:

- (a) coefficients $\mathcal{M} \in A(X)$,
- (b) variable(s) z (resp., z, y, x in the Hodge-theoretic context),
- (c) Frobenius parameters p_i (e.g., related to symmetric and alternating powers of coefficients), and
- (d) representation $V \in \text{Rep}_{\mathbb{C}}(\Sigma_n)$.

Example 2.3. (Schur functors) If $p_r = 1$ for all r , the effect is to take the Σ_n -invariant part in the Künneth formula, i.e., to compute the Betti (or Hodge) numbers of

$$H_{(c)}^*(X^n, V \otimes \mathcal{M}^{\boxtimes})^{\Sigma_n} \simeq H_{(c)}^*(S^n X, S_V(\mathcal{M})),$$

where

$$S_V(\mathcal{M}) := \left(\pi_{n*}(V \otimes \mathcal{M}^{\boxtimes}) \right)^{\Sigma_n}$$

is the *Schur power of \mathcal{M} with respect to $V \in \text{Rep}_{\mathbb{C}}(\Sigma_n)$* . These Schur-type objects $S_V(\mathcal{M})$ generalize the symmetric powers $S^n \mathcal{M}$ and alternating powers $C^n \mathcal{M}$ of \mathcal{M} (introduced in Maxim et al. (2011) and Maxim and Schürmann (2012), which correspond to the *trivial* and *sign* representations, respectively. For instance,

- (1) if $\mathcal{M} = \mathbb{C}_X \in D_c^b(X)$, then $S^n \mathbb{C}_X = \mathbb{C}_{S^n X}$.
- (2) if $\mathcal{M} = IC'_X := IC_X[-\dim_{\mathbb{C}} X] \in D_c^b(X)$, then $S^n IC'_X = IC'_{S^n X}$.
- (3) if X is projective and $\mathcal{M} = \mathcal{O}_X \in D_{coh}^b(X)$, then $S^n \mathcal{O}_X = \mathcal{O}_{S^n X}$.
- (4) if $V = V_{\mu}$ is the irreducible representation of Σ_n corresponding to the partition μ of n , then $S_{V_{\mu}}(IC'_X) \cong IC'_{S^n X}(V_{\mu})$.

Example 2.4. Assume $p_r = 1$ for all r . Then:

- (1) If $\mathcal{M} = \mathbb{C}_X \in D_c^b(X)$, Theorem 2.2(a) yields formula (1) for $P_{(c)}$, namely,

$$\sum_{n \geq 0} P_{(c)}(S^n X)(z) \cdot t^n = \exp \left(\sum_{r \geq 1} P_{(c)}(X)(z^r) \cdot \frac{t^r}{r} \right).$$

- (2) If $\mathcal{M} = IC'_X$, Theorem 2.2(a) yields an intersection cohomology version of Macdonald's formula.
- (3) If X is projective, $\mathcal{M} = \mathcal{O}_X$ and $z = 1$, Theorem 2.2(a) yields formula (3) for χ_a .
- (4) If $\mathcal{M} = \mathbb{Q}_X^H \in D^b \text{MHM}(X)$, the Hodge-theoretic counterpart of Theorem 2.2(a) yields Cheah's formula (5) for $h_{(c)}^{p,q,k}(S^n X)$.
- (5) If $\mathcal{M} = IC_X^H \in D^b \text{MHM}(X)$, the Hodge-theoretic counterpart of Theorem 2.2(a) yields an intersection cohomology version of Cheah's formula.
- (6) If X is projective, $\mathcal{M} = IC_X^H$, $x = z = 1$ and $y = -1$, the Hodge-theoretic counterpart of Theorem 2.2(a) yields an intersection cohomology version of Hirzebruch-Zagier's formula, i.e., for the Goresky-MacPherson signature of symmetric products. If, moreover, X is an orbifold, this gives Hirzebruch-Zagier's formula in the complex algebraic case.

Example 2.5. (Macdonald formula for partial quotients) Let $V = \text{Ind}_K^{\Sigma_n}(\text{triv})$ be the representation induced from the trivial representation of a subgroup K of Σ_n , and let $\mathcal{M} = \mathbb{C}_X \in D_c^b(X)$. Then Theorem 2.2(b) specializes for $p_r = 1$ (for all r) to Macdonald's Poincaré polynomial formula for the *partial quotient* X^n/K , namely:

$$P_{(c)}(X^n/K)(z) = \sum_{\lambda \vdash n} \frac{1}{z^\lambda} \chi_\lambda(\text{Ind}_K^{\Sigma_n}(\text{triv})) \cdot \prod_{r \geq 1} (P_{(c)}(X)(z^r))^{k_r}.$$

Remark 2.6. For X projective, one can recover some of the Euler characteristic-type results mentioned above by taking the degree of similar (equivariant) characteristic class formulae obtained in Maxim and Schürmann (2018).

2.5. Sketch of proof of Theorem 2.2

The proof of Theorem 2.2 is reduced via an equivariant Künneth formula to a generating series identity for abstract characters of tensor powers $\mathcal{V}^{\otimes n}$ of an element \mathcal{V} in a suitable symmetric monoidal category (A, \otimes) . Specifically, let A_{Σ_n} be the additive category of the Σ_n -equivariant objects in A , as in Maxim and Schürmann (2012, Sect. 4), with corresponding Grothendieck group $K_0(A_{\Sigma_n})$. Then

$$[\mathcal{V}^{\otimes n}] \in K_0(A_{\Sigma_n}) \simeq K_0(A) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(\Sigma_n),$$

with $\text{Rep}_{\mathbb{Q}}(\Sigma_n)$ being the ring of rational representations of Σ_n (e.g., see Maxim & Schürmann, 2012, Eqn. (45) for the above decomposition). Let cl_n be the *characteristic class* homomorphism defined by the composition:

$$K_0(A_{\Sigma_n}) \simeq K_0(A) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(\Sigma_n) \xrightarrow{id \otimes char} K_0(A) \otimes_{\mathbb{Z}} C(\Sigma_n) \xrightarrow{id \otimes ch_F} K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}[p_i, i \geq 1].$$

Then one has the following result (see Maxim & Schürmann, 2020, Theorems 1.4 and 1.5).

Theorem 2.7. (a) For any $\mathcal{V} \in A$, the following generating series identity holds:

$$\sum_{n \geq 0} cl_n([\mathcal{V}^{\otimes n}]) \cdot t^n = \exp \left(\sum_{r \geq 1} \psi_r([\mathcal{V}]) \otimes p_r \cdot \frac{t^r}{r} \right) \in (K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1])[[t]], \quad (10)$$

with ψ_r the r -th Adams operation of the pre-lambda ring $K_0(A)$.

(b) For $V \in \text{Rep}_{\mathbb{Q}}(\Sigma_n)$, the following holds in $K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1]$:

$$cl_n(V \otimes \mathcal{V}^{\otimes n}) = \sum_{\lambda \vdash n} \frac{p_\lambda}{z^\lambda} \chi_\lambda(V) \otimes \prod_{r \geq 1} (\psi_r([\mathcal{V}]))^{k_r}. \quad (11)$$

Remark 2.8. The Adams operation ψ_r of formula (10) is given by

$$\psi_r([\mathcal{V}]) := tr_r([\mathcal{V}^{\otimes r}])(\sigma_r),$$

where $\sigma_r \in \Sigma_r$ is a cycle of length r , and tr_r is the composition:

$$tr_r : K_0(A_{\Sigma_r}) \simeq K_0(A) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(\Sigma_r) \xrightarrow{id \otimes char} K_0(A) \otimes_{\mathbb{Z}} C(\Sigma_r).$$

Example 2.9. The Grothendieck ring $K_0(A)$ of the *additive* category A has the structure of a *pre-lambda ring*, i.e., it comes endowed with a family of mappings $\sigma_n : K_0(A) \rightarrow K_0(A)$ for $n \in \mathbb{N}_0$ with $\sigma_0(r) = 1$ and $\sigma_1(r) = r$ for all $r \in K_0(A)$, satisfying

$$\sigma_n(r + r') = \sum_{i=0}^n \sigma_i(r) \cdot \sigma_{n-i}(r') \quad \text{for all } n \in \mathbb{N}_0 \text{ and } r, r' \in K_0(A).$$

Then, by setting $p_r = 1$ for all r , formula (10) specializes to the well-known pre-lambda ring identity (e.g., see Knutson, 1973; or Macdonald, 1979, Ch. 1, Rem. 2.15):

$$\sigma_t([\mathcal{V}]) = 1 + \sum_{n \geq 1} [(\mathcal{V}^{\otimes n})^{\Sigma_n}] \cdot t^n = \exp \left(\sum_{r \geq 1} \psi_r([\mathcal{V}]) \cdot \frac{t^r}{r} \right) \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}[[t]], \quad (12)$$

relating the pre-lambda structure to the corresponding Adams operations.

Example 2.10. If $A = \text{Vect}_{\mathbb{Q}}$ is the category of finite dimensional rational vector spaces, then for $\mathcal{V} = \mathbb{Q}$ the unit of A with respect to the tensor product, the above formula (11) specializes to the following well-known description of the homomorphism

$$cl_n : \text{Rep}_{\mathbb{Q}}(\Sigma_n) \rightarrow \mathbb{Q}[p_i, i \geq 1]$$

given by Macdonald (1979, Sect. 7, 7.2):

$$cl_n(V) = \sum_{\lambda=(k_1, k_2, \dots) \vdash n} \frac{\chi_{\lambda}(V)}{z_{\lambda}} \cdot \prod_{r \geq 1} p_r^{k_r}.$$

The proof of Theorem 2.7 relies on Macdonald’s calculus of symmetric functions (see Macdonald, 1979). For a plethysm interpretation, see Maxim and Schürmann (2012, Theorem 1.9). Theorem 2.2 is then obtained from Theorem 2.7 by letting $\mathcal{V} = H_{(c)}^*(X, \mathcal{M})$ (or $\mathcal{V} = Gr_F^* Gr_*^W H_{(c)}^*(X, \mathcal{M})_{\mathbb{C}}$ if $\mathcal{M} \in D^b\text{MHM}(X)$), and A the abelian tensor category of finite dimensional (multi-)graded \mathbb{C} -vector spaces.

2.6. Other applications

The abstract character formula of Theorem 2.7 can also be used to derive equivariant versions of Theorem 2.2 for varieties X with additional symmetries, e.g.,

- (1) an algebraic action on X of a finite group G ,
- (2) an algebraic automorphism $g : X \rightarrow X$ of finite order,
- (3) a proper algebraic endomorphism $g : X \rightarrow X$,

and equivariant coefficients. This is done by replacing the category A by a suitable category A_G of G -equivariant objects in A , as in Maxim and Schürmann (2012), and by the

category $\text{End}(A)$ of endomorphisms of objects in A , respectively.

For example, if $\mathcal{M} = \mathbb{C}_X$ and $g : X \rightarrow X$ is a (proper) algebraic endomorphism of X , one gets an equivariant version of Macdonald's generating series formula, expressed in terms of the *graded Lefschetz Zeta function*. Specifically, one has the following (see Maxim & Schürmann, 2020, Theorem 1.7).

Theorem 2.11. *The following identity holds in $\mathbb{C}[z][[t]]$:*

$$\sum_{n \geq 0} P_{(c)}^g(S^n X)(z) \cdot t^n = \exp \left(\sum_{r \geq 1} P_{(c)}^{g^r}(X)(z^r) \cdot \frac{t^r}{r} \right), \quad (13)$$

where

$$P_{(c)}^g(X)(z) := \sum_k \text{trace}_g(H_{(c)}^k(X; \mathbb{C})) \cdot (-z)^k.$$

(A similar formula holds for the equivariant mixed Hodge polynomials.)

Formula (13) specializes for $z = 1$ to the usual Lefschetz zeta function of the (proper) endomorphism $g : X \rightarrow X$. Moreover, for $g = id_X$ the identity of X , formula (13), reduces to Macdonald's generating series formula (1), for the Poincaré polynomials and Betti numbers of the symmetric products of X .

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REFERENCES

- Baum, P., Fulton, W., & Macpherson, R. D. (1975). Riemann-Roch for singular varieties. *Publications Mathématiques de l'IHÉS*, 45, 101-145. http://www.numdam.org/item/PMIHES_1975__45__101_0/
- Borisov, L., & Libgober, A. (2002). Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex. In E. D'Hoker, D. Phong, & S.-T. Yau (Eds.), *Mirror Symmetry IV, AMS/IP Studies in Advanced Mathematics* (Vol. 33), (pp. 325-342). International Press.
- Borisov, L., & Libgober, A. (2003). Elliptic genera of singular varieties. *Duke Mathematical Journal*, 116(2), 319-351. <https://doi.org/10.1215/S0012-7094-03-11625-7>
- Brasselet, J.-P., Schürmann, J., & Yokura, S. (2010). Hirzebruch classes and motivic Chern classes for singular spaces. *Journal of Topology and Analysis*, 2(1), 1-55. <https://doi.org/10.1142/S1793525310000239>
- Bridgeland, T., King, A., & Reid, M. (2001). The McKay correspondence as an equivalence of derived categories. *Journal of the American Mathematical Society*, 14(3), 535-554. <https://doi.org/10.1090/S0894-0347-01-00368-X>

- Cappell, S., Maxim, L., Ohmoto, T., Schürmann, J., & Yokura, S. (2013). Characteristic classes of Hilbert schemes of points via symmetric products. *Geometry & Topology*, 17(2), 1165-1198. <https://doi.org/10.2140/gt.2013.17.1165>
- Cappell, S., Maxim, L., Schürmann, J., & Shaneson, J. (2012). Equivariant characteristic classes of complex algebraic varieties. *Communications on Pure and Applied Mathematics*, 65(12), 1722-1769. <https://doi.org/10.1002/cpa.21427>
- Cappell, S., Maxim, L., Shaneson, J., Schürmann, J., & Yokura, S. (2017). Characteristic classes of symmetric products of complex quasi-projective varieties. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2017(728), 35-63. <https://doi.org/10.1515/crelle-2014-0114>
- Cheah, J. (1996). On the cohomology of Hilbert schemes of points. *Journal of Algebraic Geometry*, 5(3), 479-511.
- Ellingsrud, G., Göttsche, L., & Lehn, M. (2001). On the cobordism class of the Hilbert scheme of a surface. *Journal of Algebraic Geometry*, 10(1), 81-100.
- Goresky, M., & MacPherson, R. (1980). Intersection homology theory. *Topology*, 19(2), 135-162.
- Goresky, M., & MacPherson, R. (1983). Intersection homology II. *Inventiones Mathematicae*, 72(1), 77-129.
- Göttsche, L. (1990). The Betti numbers of the Hilbert scheme of points on a smooth projective surface. *Mathematische Annalen*, 286(1-3), 193-207.
- Göttsche, L. (2001). On the motive of the Hilbert scheme of points on a surface. *Mathematical Research Letters*, 8(5-6), 613-627.
- Göttsche, L., & Soergel, W. (1993). Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. *Mathematische Annalen*, 296, 235-245.
- Gusein-Zade, S. M., Luengo, I., & Melle-Hernández, A. (2006). Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points. *Michigan Mathematical Journal*, 54(2), 353-359.
- Knutson, D. (1973). *λ -rings and the representation theory of the symmetric group: Lecture notes in mathematics* (Vol. 308). Springer-Verlag.
- Kupers, A., & Miller, J. (2016). Homological stability for topological chiral homology of completions. *Advances in Mathematics*, 292, 755-827.
- Macdonald, I. G. (1962a). The Poincaré polynomial of a symmetric product. *Proceedings of the Cambridge Philosophical Society*, 58, 563-568.
- Macdonald, I. G. (1962b). Symmetric products of an algebraic curve. *Topology*, 1(4), 319-343.
- Macdonald, I. G. (1979). *Symmetric functions and Hall polynomials: Oxford mathematical monographs*. Oxford University Press.
- MacPherson, R. D. (1974). Chern classes for singular algebraic varieties. *Annals of Mathematics*, 100(2), 423-432.
- Maulik, D., Nekrasov, N., Okounkov, A., & Pandharipande, R. (2006). Gromov-Witten theory and Donaldson-Thomas theory, I. *Compositio Mathematica*, 142(5), 1263-

1285.

- Maxim, L., Saito, M., & Schürmann, J. (2011). Symmetric products of mixed Hodge modules. *Journal de Mathématiques Pures et Appliquées*, 96(5), 462-483.
- Maxim, L., & Schürmann, J. (2012). Twisted genera of symmetric products. *Selecta Mathematica (New Series)*, 18(1), 283-317.
- Maxim, L., & Schürmann, J. (2018). Equivariant characteristic classes of external and symmetric products of varieties. *Geometry & Topology*, 22(1), 417-515.
- Maxim, L., & Schürmann, J. (2020). Plethysm and cohomology representations of external and symmetric products. *Advances in Mathematics*, 375, 107373.
- Moonen, B. (1978). *Das Lefschetz-Riemann-Roch-theorem für singuläre Varietäten* [The Lefschetz-Riemann-Roch theorem for singular varieties]. *Bonner Mathematische Schriften* (Vol. 106). Universität Bonn.
- Ohmoto, T. (2008). Generating functions of orbifold Chern classes I: Symmetric products. *Mathematical Proceedings of the Cambridge Philosophical Society*, 144(2), 423-438.
- Vakil, R., & Wood, M. (2015). Discriminants in the Grothendieck ring. *Duke Mathematical Journal*, 164(6), 1139-1185.
- Zagier, D. (1972). The Pontrjagin class of an orbit space. *Topology*, 11(3), 253-264.