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GROTHENDIECK RINGS OF DEFINABLE SUBASSIGNMENTS AND EQUIVARIANT MOTIVIC MEASURES

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Abstract

This paper studies categories of definable subassignments with some category equivalences to semi-algebraic and constructible subsets of arc spaces of algebraic varieties. These equivalences lead to the identity of certain Grothendieck rings, which allows us to compare the motivic measure of Cluckers-Loeser with that of Denef-Loeser for certain classes of definable subassignments.

Keywords: Definable subassignments; Grothendieck ring; Measurable subassignments; Motivic measure.

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1. INTRODUCTION

Since it was invented by Kontsevich at the 1995 Orsay seminar, geometric motivic integration has attained a full development and has become one of the central objects of algebraic geometry. From algebraic varieties to formal schemes, the development records the contributions of several authors, such as Denef and Loeser (1998, 1999), Loeser and Sebag (2003), Nicaise (2009), Nicaise and Sebag (2007), and Sebag (2004). Another point of view on motivic integration known as arithmetic motivic integration, which works over p-adic fields (see Denef & Loeser, 2001). Cluckers-Loeser's motivic integration (see Cluckers & Loeser, 2005, 2008, 2010), which was built on model theory with respect to the Denef-Pas languages, is a general theory of motivic integration. It allows specialization to both arithmetic and geometric points of view (see Cely & Raibaut, 2019; Cluckers et al., 2014; Gordon & Yaffe, 2009). The theory of motivic integration has an important application to the Fundamental Lemma (see Cluckers et al., 2011).

Using model theory with different languages, Hrushovski and Kazhdan (2006) and Hrushovski and Loeser (2016) also give extensions of geometric motivic integration to the arithmetic aspect, with many interesting results and applications.

Throughout the present paper, the ground field k will always be a field of characteristic zero. The paper discusses the motivically measurable subassignments in the formalism of Cluckers-Loeser for motivic integration (Cluckers & Loeser, 2005, 2008, 2010), it also provides a comparison of their measure with the classical motivic measure of Denef and Loeser (1998, 1999). For this purpose, we concentrate on a special Denef-Pas language $\mathscr{L}_{\mathrm{DP,P}}$, which includes the Presburger language for value group sort, and consider the theory T_{acl} of algebraically closed fields containing k. Let Field_k be the category of all fields K containing k and $\mathrm{Field}_k(T_{\mathrm{acl}})$ be the category of fields K over k such that each $(K((t)),K,\mathbb{Z})$ is a model of T_{acl}. For K in Field_k, we consider the natural valuation map $\mathrm{ord}_t : K((t))^{\times} \to \mathbb{Z}$ augmented by $\mathrm{ord}_t(0) = +\infty$ and the natural angular component map $\overline{\mathrm{ac}} : K((t))^{\times} \to K$ augmented by $\overline{\mathrm{ac}}(0) = 0$. A basic affine subassignment $h[m,n,r]$ (or, in another notation, $h_{\mathbb{A}^m_{k((t))}\times\mathbb{A}^n_k\times\mathbb{Z}^r}$) is a functor $K \mapsto K((t))^m \times K^n \times \mathbb{Z}^r$ from Field_k to the category of sets. A definable subassignment of $h[m,n,r]$ is a set of points in $h[m,n,r]$ satisfying a given formula φ; it is not a functor in general. In the first half of this paper, we study the categories concerning definable T_{acl}-subsassignments $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})$ and $\mathrm{SDef}_k(X,\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})$ in which the language $\mathscr{L}_{\mathrm{DP,P}}(k)$ is an extension of $\mathscr{L}_{\mathrm{DP,P}}$ made by adding constants in k so that all polynomials in both valued field sort and residue field sort have coefficients in k.

The category $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})$ has objects that are small definable T_{acl}-subassi -gnments comparable with the category SA_k of semi-algebraic subsets of the arc space of an algebraic k-variety. When fixing a k-variety X, we get the subcategory $\mathrm{SDef}_k(X, \mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})$ of $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k))$ whose objects are all the small definable T_{acl}-subassig -nments of $h_{X\times_k \mathrm{Spec}k((t))}$. The definitions of SA_k and $\mathrm{SA}_k(X)$ are given in Section 3.1. The first main result of this paper is as follows.

Theorem 1 (Theorem 4.1)**.** *The categories* $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* SA_k *are equivalent. If X is an algebraic k-variety, the categories* $\mathrm{SDef}_k(X, \mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* $\mathrm{SA}_k(X)$ *are equivalent.*

Let S be an affine k-variety, and let $\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ be the category whose objects $\mathsf{X} \to h_S$ are the h_S-projection of definable T_{acl}-subassignment X of $h_S \times h_{\mathbb{A}_k^n} = h_{S\times_k \mathbb{A}_k^n}$ for some n in $\mathbb{N}$. In Section 2 we describe the category Cons_S of constructible morphisms from constructible sets over k to S in which a morphism in Cons_S from $X \to S$ to $Y \to S$ is determined uniquely up to an S-isomorphism on X by the graph of an S-morphism $X \to Y$. The category Cons_S and Var_S have the same Grothendieck ring. The second main result of this paper is as follows.

Theorem 2 (Theorem 4.2)**.** *For any k-variety S, the categories* $\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* Cons_S *are equivalent.*

This theorem has several interesting corollaries, such as the following isomorphism between Grothendieck rings $K_0(\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \cong K_0(\mathrm{Var}_S)$. Due to this isomorphism we can identify the class $\mathbb{L}$ of the trivial line bundle $S \times_k \mathbb{A}_k^1 \to S$ with the class $\left[h_{S\times_k \mathbb{A}_k^1} \to h_S\right]$. Moreover, if we put

$$\mathbb{A} := \mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, \frac{1}{1-\mathbb{L}^{-n}} \mid n \in \mathbb{N}^*\right],$$

we have $K_0(\mathrm{RDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A} \cong \mathcal{M}_{\mathrm{loc}}$. We also obtain the monodromic versions $K_0^{\hat{\mu}}(\mathrm{RDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \cong K_0^{\hat{\mu}}(\mathrm{Var}_k)$ and $K_0^{\hat{\mu}}(\mathrm{RDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A} \cong \mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. Here, $\mathcal{M}_{\mathrm{loc}}$ (and $\mathcal{M}_{\mathrm{loc}}^G$, with G being either the group scheme μ_e or $\hat{\mu} = \varprojlim_e \mu_e$) are the localizations of $K_0(\mathrm{Var}_k)$ (and $K_0^G(\mathrm{Var}_k)$, respectively) obtained by inverting $\mathbb{L}$ and $\mathbb{L}^n - 1$ for all n in $\mathbb{N}^*$.

Theorem 10.1.1 of Cluckers and Loeser (2008) implies that there is a unique functor from the category of definable subassignments to the category of abelian groups, $\mathsf{X} \mapsto \mathrm{I}C(\mathsf{X})$, which assigns to $\mathsf{X} \to h_{\mathrm{Spec}k}$ a group morphism

$$\mu : \mathrm{I}C(\mathsf{X}) \to \mathcal{M}_{\mathrm{loc}}$$

satisfying axioms A0-A8 in that theorem. By Cluckers and Loeser (2008, Proposition 12.2.2), if a definable subassignment X of $h[m,n,0]$ is bounded (see Section 5.2), the characteristic function $\mathbf{1}_{\mathsf{X}}$ will be in $\mathrm{I}C(\mathsf{X})$. In this case, $\mu(\mathsf{X}) := \mu(\mathbf{1}_{\mathsf{X}})$ in $\mathcal{M}_{\mathrm{loc}}$ is the motivic measure of X. When X is an invariant positively bounded definable subassignment of $h[m,n,0]$, we obtain the following comparison theorem (which also contains the main results of the present paper).

Theorem 3 (Theorem 5.4, Proposition 5.6)**.** *Let* X *be an invariant definable subassignment of* $h[m,n,0]$ *such that, for every* (x,y) *on* X *with* $x = (x_1, \ldots, x_m)$, $\mathrm{ord}_t x_i \geq 0$ *for all*

$1 \leq i \leq m$. *With the notions of the morphism* loc *defined in Section 2,* vol *in Lemma 5.3, and* $\mathsf{X}[e]$ *in the paragraph before Proposition 5.5, for* $e \in \mathbb{N}^*$*, the following identities hold:*

$$\mu(\mathsf{X}) = \mathrm{loc}(\mathrm{vol}(\mathsf{X})) \quad \textit{in } \mathcal{M}_{\mathrm{loc}},$$
$$\mu(\mathsf{X}[e]) = \mathrm{loc}(\mathrm{vol}(\mathsf{X}[e])) \quad \textit{in } \mathcal{M}_{\mathrm{loc}}^{\mu_e}.$$

In fact, for X small in $h[m,0,0]$, Cluckers and Loeser (2008) showed that $\delta(\mu(\mathsf{X})) = \mu'(X)$ in $\widehat{\mathcal{M}}_k$, where $\widehat{\mathcal{M}}_k$ is a completion of $\mathcal{M}_k$, as defined in Denef and Loeser, 1999, δ is the canonical morphism $\mathcal{M}_{\mathrm{loc}} \to \widehat{\mathcal{M}}_k$, μ' is the Denef-Loeser motivic volume defined in Denef and Loeser (1999), and X is the semi-algebraic subset of $\mathcal{L}(\mathbb{A}_k^m)$ corresponding to X via the equivalence of categories between $\mathrm{SA}_k(\mathbb{A}_k^m)$ and $\mathrm{SDef}_k(\mathbb{A}_k^m, \mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ in Theorem 4.1.

At the end of this paper, we give a proof of the rationality of the series $\sum_{e \in \mathbb{N}^*} \mu(\mathsf{X}[e]) T^e$ in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}[[T]]$ with X an invariant positively definable subassignment of $h[m,n,0]$.

2. GROTHENDIECK RINGS OF VARIETIES

Let k be a field of characteristic zero, and let S be an algebraic k-variety. As usual (see Denef & Loeser, 1998, 1999), denote by Var_S the category of S-varieties and $K_0(\mathrm{Var}_S)$ by its Grothendieck ring. By definition, $K_0(\mathrm{Var}_S)$ is the quotient of the free abelian group generated by the S-isomorphism classes $[X \to S]$ in Var_S modulo the following relation

$$[X \to S] = [Y \to S] + [X \setminus Y \to S]$$

for Y being Zariski closed in X. Together with the fiber product over S, $K_0(\mathrm{Var}_X)$ is a commutative ring with unity $1 = [\mathrm{Id} : S \to S]$. Put

$$\mathbb{L} = [\mathbb{A}_k^1 \times_k S \to S]$$

and write $\mathcal{M}_S$ for the localization of $K_0(\mathrm{Var}_S)$ inverting $\mathbb{L}$. Denote by $\mathcal{M}_{S,\mathrm{loc}}$ the localization of $\mathcal{M}_S$ inverting $\mathbb{L}^n - 1$ for all n in $\mathbb{N}^*$.

Let $C\mathrm{Var}_S$ be the category whose objects are constructible morphisms from constructible sets over k to S with the set of morphisms between objects $X \to S$ and $Y \to S$ given by

$$\mathrm{Mor}_{C\mathrm{Var}_S}(X \to S, Y \to S) := K_0(\mathrm{Var}_{X \times_S Y}).$$

In other words, a morphism from $X \to S$ to $Y \to S$ in $C\mathrm{Var}_S$ is a finite sum of elements of the form $[U \to X \times_S Y]$ in $K_0(\mathrm{Var}_{X \times_S Y})$, with U being an algebraic k-variety. The composition of two basic morphisms $[U \to X \times_S Y]$ and $[V \to Y \times_S Z]$ is the following morphism

$$[V \to Y \times_S Z] \circ [U \to X \times_S Y] := [U \times_Y V \to X \times_S Z].$$

This definition makes sense since the morphism $U \times_Y V \to X \times_S Z$ commutes with the structural morphisms to S, and it can also be extended by additivity. Clearly, the identity morphism of X in $C\mathrm{Var}_k$ is the class in $K_0(\mathrm{Var}_{X \times_k X})$ of the diagonal morphism

$X \to X \times_k X$.

Denote by Cons_S the subcategory of $C\mathrm{Var}_S$ in which objects of Cons_S are objects of $C\mathrm{Var}_S$ and a morphism of Cons_S from $X \to S$ to $Y \to S$ is an element

$$[(\mathrm{Id}_X, f) : X \to X \times_S Y]$$

in $K_0(\mathrm{Var}_{X\times_S Y})$. By definition, each morphism of Cons_S from $X \to S$ to $Y \to S$ is determined uniquely, up to S-automorphism on X, by the constructible S-morphism of constructible sets $f : X \to Y$, or alternatively, by the graph of such an f. Using the above definition of Grothendieck ring for the category Cons_S we get

$$K_0(\mathrm{Var}_S) \cong K_0(\mathrm{Cons}_S).$$

Let X be an algebraic k-variety, and let G be an algebraic group that acts on X. The G-action is called *good* if every G-orbit is contained in an affine open subset of X. Now we fix a good action of G on the k-variety S. By definition, the G-equivariant Grothendieck group $K_0^G(\mathrm{Var}_S)$ of G-equivariant morphisms of k-varieties $X \to S$, where X is endowed with a good G-action, is the quotient of the free abelian group generated by the G-equivariant isomorphism classes $[X \to S, \sigma]$ modulo the following relations

$$[X \to S, \sigma] = [Y \to S, \sigma|_Y] + [X \setminus Y \to S, \sigma|_{X\setminus Y}]$$

for Y being σ-stable Zariski closed in X, and

$$[X \times_k \mathbb{A}_k^n \to S, \sigma] = [X \times_k \mathbb{A}_k^n \to S, \sigma']$$

if σ and σ' lift the same G-action on X to an affine action on $X \times \mathbb{A}_k^n$. As above, we have the commutative ring with unity structure on $K_0^G(\mathrm{Var}_S)$ by fiber product, where the G-action on the fiber product is through the diagonal G-action, and we may define the localization $\mathcal{M}_S^G$ of the ring $K_0^G(\mathrm{Var}_S)$ by inverting $\mathbb{L}$. In this article, we also consider the localization $\mathcal{M}_{S,\mathrm{loc}}^G$ of $\mathcal{M}_S^G$ with respect to the multiplicative family generated by the elements $1 - \mathbb{L}^{-n}$ with n in $\mathbb{N}^*$.

Since a constructible subset X of a k-variety is a finite disjoint union of locally closed subsets, we can endow X with good G-action via its locally closed subsets. A constructible morphism is G-equivariant if its graph admits a good G-action induced from the actions on its source and target. So we can define categories $C\mathrm{Var}_S^G$ and Cons_S^G as follows. As above, fix a good G-action on the k-variety S. Objects of $C\mathrm{Var}_S^G$ are G-equivariant constructible morphisms from constructible sets endowed with a good G-action to S (over k), and the set of morphisms between objects $X \to S$ and $Y \to S$ is

$$\mathrm{Mor}_{C\mathrm{Var}_S^G}(X \to S, Y \to S) := K_0(\mathrm{Var}_{X\times_S Y}^G).$$

Objects of Cons_S^G are objects of $C\mathrm{Var}_S^G$, and a morphism of Cons_S^G from $X \to S$ to $Y \to S$ is

$$[(\mathrm{Id}, f) : X \to X \times_S Y],$$

where $f : X \to Y$ is a G-equivariant constructible S-morphism of constructible sets. As before, we can define the G-equivariant Grothendieck ring $K_0(\mathrm{Cons}_S^G)$ in the usual way, and obtain a canonical isomorphism of rings

$$K_0^G(\mathrm{Var}_S) \cong K_0(\mathrm{Cons}_S^G).$$

Let $\hat{\mu}$ be the group scheme of roots of unity, which is the projective limit of group schemes $\mu_n = \mathrm{Spec}k[t]/(t^n-1)$ together with transitions $\mu_{mn} \to \mu_n$ induced by $\lambda \mapsto \lambda^m$. A good $\hat{\mu}$-action on an S-variety X is a good μ_n-action on the S-variety X for some n in $\mathbb{N}^*$. We define

$$K_0^{\hat{\mu}}(\mathrm{Var}_S) = \varinjlim K_0^{\mu_n}(\mathrm{Var}_S), \quad \mathcal{M}_S^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_S)\left[\mathbb{L}^{-1}\right],$$

and

$$\mathcal{M}_{S,\mathrm{loc}}^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_S)\left[\mathbb{L}^{-1}, (\mathbb{L}^n-1)^{-1}\right]_{n\in\mathbb{N}^*}.$$

Clearly, we have the identities

$$\mathcal{M}_S^{\hat{\mu}} = \varinjlim \mathcal{M}_S^{\mu_n} \quad \text{and} \quad \mathcal{M}_{S,\mathrm{loc}}^{\hat{\mu}} = \varinjlim \mathcal{M}_{S,\mathrm{loc}}^{\mu_n}.$$

By abuse of notation, we shall write loc for any of the following localization morphisms $\mathcal{M}_S \to \mathcal{M}_{S,\mathrm{loc}}$, $\mathcal{M}_S^{\mu_n} \to \mathcal{M}_{S,\mathrm{loc}}^{\mu_n}$, and $\mathcal{M}_S^{\hat{\mu}} \to \mathcal{M}_{S,\mathrm{loc}}^{\hat{\mu}}$ in the present paper.

When S is $\mathrm{Spec}k$, we shall write simply Var_k, $\mathcal{M}_k$, $\mathcal{M}_k^G$, $\mathcal{M}_{\mathrm{loc}}$ and $\mathcal{M}_{\mathrm{loc}}^G$ instead of $\mathrm{Var}_{\mathrm{Spec}k}$, $\mathcal{M}_{\mathrm{Spec}k}$, $\mathcal{M}_{\mathrm{Spec}k}^G$, $\mathcal{M}_{\mathrm{Spec}k,\mathrm{loc}}$ and $\mathcal{M}_{\mathrm{Spec}k,\mathrm{loc}}^G$, respectively.

3. ARC SPACES AND RATIONAL SERIES

3.1. Arc spaces

Let X be an algebraic k-variety. For $e \in \mathbb{N}^*$, let $\mathcal{L}_e(X)$ be the space of e-jet schemes of X, which is actually a k-scheme representing the functor sending a k-algebra A to the set of morphisms of k-schemes

$$\mathrm{Spec}(A[t]/(t^{e+1})) \to X.$$

Thus, the set of A-rational points of $\mathcal{L}_e(X)$ is naturally identified with the set of $A[t]/(t^{e+1})$-rational points of X.

For $d \geq e$ in $\mathbb{N}^*$, the truncation modulo t^{e+1} induces an affine morphism of k-schemes

$$\mathcal{L}_d(X) \to \mathcal{L}_e(X)$$

denoted by π_e^d. If X is a smooth variety of dimension d, the morphism π_e^d is a locally trivial fibration with fiber $\mathbb{A}_k^{(d-e)\dim_k X}$.

The above jet schemes $\mathcal{L}_e(X)$ and truncation morphisms π_e^d form a projective system of k-schemes in a natural way. As the truncation morphisms are affine, the projective limit of this system exists in the category of k-schemes and is called an *arc space of* X and denoted by $\mathcal{L}(X)$ with truncation morphisms

$$\pi_e : \mathcal{L}(X) \to \mathcal{L}_e(X).$$

If $k \subseteq K$ is a field extension of k, then the K-rational points of $\mathcal{L}(X)$ correspond one-to-one to the $K[[t]]$-rational points of X.

Recall from Denef and Loeser (1999, Section 2) that for any algebraically closed field K containing k, a subset of $K((t))^m \times \mathbb{Z}^r$ is *semi-algebraic* if it is a finite boolean combination of sets of the forms

$$\{(x,\alpha) \in K((t))^m \times \mathbb{Z}^r \mid \mathrm{ord}_t f(x) \geq \mathrm{ord}_t g(x) + \ell(\alpha)\}, \tag{1}$$

and

$$\{(x,\alpha) \in K((t))^m \times \mathbb{Z}^r \mid \mathrm{ord}_t f(x) \equiv \ell(\alpha) \mod n\}, \tag{2}$$

and

$$\left\{(x,\alpha) \in K((t))^m \times \mathbb{Z}^r \mid \Phi(\overline{\mathrm{ac}}(f_1(x)),\ldots,\overline{\mathrm{ac}}(f_p(x))) = 0\right\}, \tag{3}$$

where f, g, f_i and Φ are k-polynomials, ℓ is a $\mathbb{Z}$-polynomial of degree at most 1, n is in $\mathbb{N}$, and $\overline{\mathrm{ac}}(f_i(x))$ is the angular component of $f_i(x)$. One calls a collection of formulas defining a semi-algebraic set a *semi-algebraic condition*. A subset A of $\mathcal{L}(X)$ is called *semi-algebraic* if there exists a covering of X by affine Zariski open sets U such that $A \cap \mathcal{L}(U)$ is of the form

$$A \cap \mathcal{L}(U) = \{x \in \mathcal{L}(U) \mid \theta(f_1(\tilde{x}),\ldots,f_p(\tilde{x});\alpha)\}, \tag{4}$$

where f_i are regular functions on U, θ is a semi-algebraic condition, α may be a given tuple of integers or nothing, and $\tilde{x}$ is the element in $\mathcal{L}(U)(k(x))$ corresponding to a point x in $\mathcal{L}(U)$ of residue field $k(x)$.

By Pas (1989), if $g : X \to Y$ is a morphism of algebraic k-varieties and A is a semi-algebraic subset of $\mathcal{L}(X)$, then $g(A)$ is a semi-algebraic subset of Y. Then the map $g : A \to g(A)$ is called a *semi-algebraic morphism* of semi-algebraic sets. More generally, let A and B be semi-algebraic subsets of $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, the arc spaces of k-varieties X and Y, respectively, and let $h : A \to B$ be a map. Then h is called a *semi-algebraic morphism* if its graph is a semi-algebraic subset of $\mathcal{L}(X \times_k Y)$. Denote by SA_k the category whose objects are pairs $(A,\mathcal{L}(X))$, where A is a semi-algebraic subset of the arc space $\mathcal{L}(X)$ of an algebraic k-variety X, and a morphism of SA_k between two objects $(A,\mathcal{L}(X))$ and $(B,\mathcal{L}(Y))$ is a semi-algebraic morphism of semi-algebraic sets $A \to B$. For a given k-variety X, we can consider the full subcategory $\mathrm{SA}_k(X)$ of SA_k consisting of semi-algebraic subsets of $\mathcal{L}(X)$.

In the sense of Denef and Loeser (1999, Definition-Proposition 3.2), Denef-Loeser's motivic volume is defined on $\mathrm{ObSA}_k(X)$ as the set of all the semi-algebraic subsets of $\mathcal{L}(X)$ with reasonable properties. By Cluckers and Loeser (2008, Remark 16.3.2), this motivic volume essentially takes values in $\mathcal{M}_{\mathrm{loc}}$. In the present article, we denote Denef-Loeser's motivic volume by μ'; the symbol μ will devote Cluckers-Loeser's motivic volume, as in Cluckers and Loeser (2008).

3.2. Rationality

Let $\mathcal{M}$ be a commutative ring with unity containing $\mathbb{L}$ and $\mathbb{L}^{-1}$, and let $\mathcal{M}[[T]]$ be the set of formal power series in T with coefficients in $\mathcal{M}$, which is a ring and also an $\mathcal{M}$-module with respect to the usual operations for series. Denote by $\mathcal{M}[[T]]_{\mathrm{sr}}$ the submodule of $\mathcal{M}[[T]]$ generated by 1 and by finite products of terms

$$\frac{\mathbb{L}^p T^q}{(1-\mathbb{L}^p T^q)}$$

for (p,q) in $\mathbb{Z}\times\mathbb{N}^*$. An element of $\mathcal{M}[[T]]_{\mathrm{sr}}$ is called a *rational* series. By Denef and Loeser (1998), there exists a unique $\mathcal{M}$-linear morphism

$$\lim_{T\to\infty} : \mathcal{M}[[T]]_{\mathrm{sr}} \to \mathcal{M}$$

such that for any (p,q) in $\mathbb{Z}\times\mathbb{N}^*$,

$$\lim_{T\to\infty} \frac{\mathbb{L}^p T^q}{(1-\mathbb{L}^p T^q)} = -1.$$

Let us recall some examples of rationality. Let X be a smooth algebraic k-variety of pure dimension m, and f a regular function on X with zero locus $X_0 \neq \emptyset$. For e in $\mathbb{N}^*$, put

$$X[e] = \left\{\gamma \in \mathcal{L}_e(X) \mid f(\gamma) = t^e \mod t^{e+1}\right\},$$

which is naturally an X_0-variety and stable under the action $\lambda\cdot\gamma(t) := \gamma(\lambda t)$ of μ_e on $\mathcal{L}_e(X)$. Write simply $\left[X[e]\right]$ for the class $\left[X[e]\to X_0\right]$ in $\mathcal{M}_{X_0}^{\mu_e}$. It is proved in in Denef and Loeser (1998), that the series

$$Z_f(T) := \sum_{e\in\mathbb{N}^*} \left[X[e]\right]\mathbb{L}^{-em}T^e,$$

is a rational series, i.e., in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]_{\mathrm{sr}}$. More generally, we can obtain the rationality of a series generalizing $Z_f(T)$ without assuming that X is smooth, and with f concerning several semi-algebraic subsets in $\mathcal{L}(X)$. Let μ' be Denef-Loeser's motivic volume defined in Denef and Loeser (1999). The following theorem is a result given in Lê and Nguyen (2020, Proposition 4.6).

Theorem 3.1 (Lê & Nguyen, 2020)**.** *Let X be a k-variety and f a regular function on X. Let A_α, α in $\mathbb{N}^r$, be a family of semi-algebraic subsets of $\mathcal{L}(X)$ such that there exists a covering of X by affine Zariski open sets U satisfying the condition that $A_\alpha \cap \mathcal{L}(U)$ are finite boolean combinations of sets of the forms (1) and (2). Assume that, for every α in $\mathbb{N}^r$, A_α is stable in the sense of Denef and Loeser (1999) and disjoint with $\mathcal{L}(X_{Sing})$. For $e \in \mathbb{N}^*$, we put*

$$A_{e,\alpha} := \left\{\gamma \in A_\alpha \mid f(\gamma) = t^e \mod t^{e+1}\right\}.$$

Let Δ be a rational polyhedral convex cone in $\mathbb{R}^{r+1}_{\geq 0}$ and $\bar{\Delta}$ its closure. Let ℓ and ℓ' be integral linear forms on $\mathbb{Z}^{r+1}$ with $\ell(e,\alpha) > 0$ and $\ell'(e,\alpha) \geq 0$ for all (e,α) in $\bar{\Delta} \setminus \{0\}$. Then the formal power series

$$Z(T) := \sum_{(e,\alpha)\in\Delta\cap\mathbb{N}^{r+1}} \mu'(A_{e,\alpha})\,\mathbb{L}^{-\ell'(e,\alpha)} T^{\ell(e,\alpha)}$$

is an element of $\mathcal{M}_k^{\hat{\mu}}[[T]]_{\mathrm{sr}}$, and the limit $\lim_{T\to\infty} Z(T)$ is independent of such an ℓ and ℓ'.

4. CATEGORIES OF DEFINABLE SUBASSIGNMENTS

In this section, we shall recall some concepts and results on motivic integration in the sense of Cluckers and Loeser (2008). We also provide an equivariant version concerning definable T_{acl}-subassignments, where T_{acl} is the theory of all algebraically closed fields containing k.

4.1. Definable subassignments

We consider the formalism of Cluckers and Loeser (2008) with a concrete Denef-Pas language $\mathscr{L}_{\mathrm{DP,P}}$ consisting of the ring language $\mathbf{L}_{\mathrm{Rings}} = \{+,-,\cdot,0,1\}$ for valued fields, the ring language $\mathbf{L}_{\mathrm{Rings}}$ for residue fields, and the Presburger language $\mathbf{L}_{\mathrm{PR}}$ for value groups, where

$$\mathbf{L}_{\mathrm{PR}} = \{+,-,0,1,\leq\} \cup \{\equiv_n \mid n \in \mathbb{N}^*\},$$

and $\equiv_n$ is the equivalence relation modulo n.

Let Field_k be the category of all fields K containing k whose morphisms are field morphisms. For any K in Field_k, we consider the natural valuation map $\mathrm{ord}_t : K((t))^\times \to \mathbb{Z}$ augmented by $\mathrm{ord}_t(0) = +\infty$, and the natural angular component map $\overline{\mathrm{ac}} : K((t))^\times \to K$ augmented by $\overline{\mathrm{ac}}(0) = 0$.

For a basic set

$$V := \mathbb{A}^m_{k((t))} \times \mathbb{A}^n_k \times \mathbb{Z}^r,$$

with m, n, r in $\mathbb{N}$, we consider the functor h_V (also denoted by $h[m,n,r]$) from Field_k to the category of sets defined by

$$h_V(K) = h[m,n,r](K) := K((t))^m \times K^n \times \mathbb{Z}^r.$$

If X is a map sending each object K of Field_k to a subset $\mathsf{X}(K)$ of $K((t))^m \times K^n \times \mathbb{Z}^r$, then X is called an *affine subassignment* (or, briefly, *subassignment*) of $h_V = h[m,n,r]$. Note that X is not necessarily a subfunctor of $h[m,n,r]$. In the same way, we can define morphisms of subassignments and their graphs, as well as the union, subtraction, Cartesian product and fiber product of two subassignments.

A subassignment X of $h[m,n,r]$ is called *definable* if there exists a formula φ in $\mathscr{L}_{\mathrm{DP,P}}$ with $k((t))$-coefficients and m free variables in the valued field sort, k-coefficients and n free variables in residue field sort, and r free variables in the value group sort, such that, for any K in Field_k,

$$\mathsf{X}(K) = \{x \in K((t))^m \times K^n \times \mathbb{Z}^r \mid (K((t)), K, \mathbb{Z}) \models \varphi(x)\}.$$

In this setting, we also write h_φ for the definable subassignment X. Denote by $\emptyset$ the empty definable subassignment with $\emptyset(K) = \emptyset$ for any K in Field_k. For X and X' being definable subassignments of $h[m,n,r]$ and $h[m',n',r']$, respectively, a *definable* morphism $\mathsf{X} \to \mathsf{X}'$ is a morphism of subassignments $\mathsf{X} \to \mathsf{X}'$ such that its graph is a definable subassignment of $h[m+m', n+n', r+r']$.

For a set

$$W := \mathscr{X} \times X \times \mathbb{Z}^r,$$

with $\mathscr{X}$ an algebraic $k((t))$-variety, and X an algebraic k-variety, we define

$$h_W(K) := \mathscr{X}(K((t))) \times X(K) \times \mathbb{Z}^r,$$

for any K in Field_k. In general, we can define definable subassignments of h_W, definable morphisms of definable subassignments, and the usual operations on definable subassignments of functors of the form h_W using a glueing procedure, as in Cluckers and Loeser (2008, Section 2.3). We take finite covers (which always exist) of $\mathscr{X}$ and X by affine open $k((t))$-subvarieties and k-subvarieties, respectively, then go back to the definition of affine definable subassignment and glue them.

We consider the category $\mathrm{Def}_k(\mathscr{L}_{\mathrm{DP,P}})$ (or Def_k for short) of affine definable subassignments, where its objects are pairs $(\mathsf{X}, h[m,n,r])$, X is a definable subassignment of $h[m,n,r]$ and a morphism

$$(\mathsf{X}, h[m,n,r]) \to (\mathsf{X}', h[m',n',r'])$$

in Def_k is a definable morphism $\mathsf{X} \to \mathsf{X}'$. We also consider the category $\mathrm{GDef}_k(\mathscr{L}_{\mathrm{DP,P}})$ (or GDef_k for short) of global definable subassignments, where objects of $\mathrm{GDef}_k(\mathscr{L}_{\mathrm{DP,P}})$ are pairs (X, h_W) with h_W as above and X being a definable subassignment of h_W, and a morphism

$$(\mathsf{X}, h_W) \to (\mathsf{X}', h_{W'})$$

in GDef_k is a definable morphism $\mathsf{X} \to \mathsf{X}'$. For any affine definable subassignment S, we denote by $\mathrm{Def}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}})$ (or $\mathrm{Def}_{\mathsf{S}}$ for short) the category of morphisms $\mathsf{X} \to \mathsf{S}$ in Def_k, and a morphism in $\mathrm{Def}_{\mathsf{S}}$ between $\mathsf{X} \to \mathsf{S}$ and $\mathsf{X}' \to \mathsf{S}$ as a morphism $\mathsf{X} \to \mathsf{X}'$ in Def_k that is compatible with the morphisms to S. For any definable subassignment S, the category $\mathrm{GDef}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}})$ (or $\mathrm{GDef}_{\mathsf{S}}$ for short) can be defined in the same way as GDef_k with $h_{\mathrm{Spec}k}$ replaced by S.

When we consider the category $\mathrm{Field}_k(T_{\mathrm{acl}})$ of all algebraically closed fields containing k in stead of Field_k and use the language $\mathscr{L}_{\mathrm{DP,P}}(k)$ (see the definition in Section 4.3) instead of $\mathscr{L}_{\mathrm{DP,P}}$, we obtain the corresponding notions of T_{acl}-subassignment, $\mathrm{Def}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ and $\mathrm{GDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$.

4.2. Points on definable subassignments

Let X be an object in GDef_k. A point x on X is a tuple $x = (x_0, K)$ such that K is in Field_k and x_0 is in $\mathsf{X}(K)$. For such a point x on X we usually write $k(x)$ for K and call it the residue field of x. Let

$$f : \mathsf{X} \to \mathsf{Y}$$

be a morphism in Def_k, with

$$\mathsf{X} = (\mathsf{X}_0, h[m,n,r])$$

and

$$\mathsf{Y} = (\mathsf{Y}_0, h[m',n',r']),$$

whose graph is defined by a formula $\varphi(x,y)$, where x is in $h[m,n,r]$ and y is in $h[m',n',r']$. One defines the fiber of f over a point $y = (y_0, k(y))$ on Y to be the definable subassignment X_y in $\mathrm{Def}_{k(y)}$ given by the formula $\varphi(x, y_0)$. In the category GDef_k, fibers of a morphism are defined in the same way by using affine covers.

4.3. Categories $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ and $\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$

Denote by $\mathscr{L}_{\mathrm{DP,P}}(k)$ the language extending $\mathscr{L}_{\mathrm{DP,P}}$ by adding constants in k so that all polynomials in both the valued field sort and residue field sort have coefficients in k. Let X be an algebraic k-variety, let

$$\mathscr{X} := X \times_k \mathrm{Spec}k((t)),$$

and let A be a definable subassignment of $h_{\mathscr{X}}$ defined by a formula in $\mathscr{L}_{\mathrm{DP,P}}(k)$. Assume that $\mathscr{X}$ is a closed subscheme in $\mathbb{A}^m_{k((t))}$, for some m in $\mathbb{N}$, such that the ideal defining $\mathscr{X}$ is generated by polynomials with coefficients in $k[[t]]$. The above-mentioned definable subassignment A (i.e., defined by a formula in $\mathscr{L}_{\mathrm{DP,P}}(k)$) is called *small* if A is contained in the following definable subassignment

$$\{(x_1, \ldots, x_m) \in h[m,0,0] \mid \mathrm{ord}_t x_i \geq 0, 1 \leq i \leq m\}.$$

For $\mathscr{X}$ not necessarily affine, we call A *small* if there exists a cover of $\mathscr{X}$ by open affine $k((t))$-subvarieties $\mathscr{U}_i$ defined by the vanishing of polynomials with coefficients in $k[[t]]$

such that $\mathsf{A} \cap h_{\mathscr{U}_i}$ are small for all i. Let $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ be the subcategory of $\mathrm{GDef}_k(\mathscr{L}_{\mathrm{DP,P}}, T_{\mathrm{acl}})$ whose objects are pairs

$$(\mathsf{A}, h_{X \times_k \mathrm{Spec}k((t))}),$$

where X is an algebraic k-variety and A is a small definable T_{acl}-subassignment of $h_{X \times_k \mathrm{Spec}k((t))}$. A morphism in $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ between objects

$$(\mathsf{A}, h_{X \times_k \mathrm{Spec}k((t))})$$

and

$$(\mathsf{B}, h_{Y \times_k \mathrm{Spec}k((t))})$$

is a T_{acl}-morphism of T_{acl}-subassignments $\mathsf{A} \to \mathsf{B}$ such that its graph is a small definable T_{acl}-subassignment of $h_{X \times_k Y \times_k \mathrm{Spec}k((t))}$.

Fixing an algebraic k-variety X, we define a category denoted by $\mathrm{SDef}_k(X, \mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$, which is the full subcategory of $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$, whose objects contain all small definable T_{acl}-subassignments of $h_{X \times_k \mathrm{Spec}k((t))}$.

Theorem 4.1. *The categories* $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* SA_k *are equivalent. If X is an algebraic k-variety, then the categories* $\mathrm{SDef}_k(X, \mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* $\mathrm{SA}_k(X)$ *are equivalent.*

Proof.

For short, we now write (A, X) instead of $(\mathsf{A}, h_{X \times_k \mathrm{Spec}k((t))})$ for an object of SDef_k $(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$, and (A, X) instead of $(A, \mathcal{L}(X))$ for an object of SA_k.

First, let us construct a functor $\mathscr{F}$ from $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ to SA_k. Let (A, X) be an object of $\mathrm{SDef}_k(X, \mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$. Then, there is a cover of $X \times_k \mathrm{Spec}k((t))$ by Zariski open affine $k((t))$-subvarieties $\mathscr{U}$, with $\mathscr{U}$ embedded as a closed $k((t))$-subvariety in some $\mathbb{A}^m_{k((t))}$. (We can take m common for all $\mathscr{U}$.) The embedding defined over $k[[t]]$ is such that for the standard coordinates x_i of $h[m,0,0]$ and any point x on $\mathsf{A} \cap h_{\mathscr{U}}$, we have $\mathrm{ord}_t x_i(x) \geq 0$. Moreover, $\mathsf{A} \cap h_{\mathscr{U}}$ is defined by a formula $\varphi(\underline{x}, \alpha)$ in the language $\mathscr{L}_{\mathrm{DP,P}}(k)$, where $\underline{x} = (x_1, \ldots, x_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_r)$ are free variables in value group sort, namely,

$$\mathsf{A} \cap h_{\mathscr{U}} = \{x \in h_{\mathscr{U}} \mid \varphi(x_1(x), \ldots, x_m(x), \alpha)\}.$$

By Denef-Pas's quantifier elimination for algebraically closed fields (see Cluckers & Loeser, 2008, Corollary 2.1.2), $\varphi(\underline{x}, \alpha)$ is equivalent to a finite disjunction of formulas of the form

$$\psi(\overline{\mathrm{ac}}g_1(\underline{x}), \ldots, \overline{\mathrm{ac}}g_q(\underline{x})) \wedge \vartheta(\mathrm{ord}_t f_1(\underline{x}), \ldots, \mathrm{ord}_t f_p(\underline{x}), \alpha), \tag{5}$$

where f_i and g_j are polynomials over k, ψ is an $\mathbf{L}_{\mathrm{Rings}}$-formula with coefficients in k, and ϑ is an $\mathbf{L}_{\mathrm{PR}}$-formula. Thus, as seen in (1), (2), and (3), the formula φ is nothing but a semi-algebraic condition.

Since all polynomials f_i and g_j have coefficients in k, and in particular, polynomials defining $\mathscr{U}$ have coefficients in k, there exists a unique closed k-subvariety U in $\mathbb{A}_k^m$ such that

$$\mathscr{U} = U \times_k \operatorname{Spec} k((t)). \tag{6}$$

Hence we have a cover $\{U\}_U$ of X by Zariski open affine k-subvarieties. The above x_i induce regular functions x'_i on U such that x'_i are standard coordinate components in $\mathbb{A}_k^m$ for every $1 \leq i \leq n$, and U is defined by the vanishing of x'_i for $n+1 \leq i \leq m$. Now we put

$$A_U := \left\{ x \in \mathcal{L}(U) \mid \varphi(x'_1(\tilde{x}), \ldots, x'_m(\tilde{x}), \alpha) \right\},$$

where $\tilde{x}$ is defined after (4), and glue the A_U's into a semi-algebraic subset A of $\mathcal{L}(X)$. Note that the construction of A is up to semi-algebraic isomorphism independent of the choice of the cover $\{\mathscr{U}\}_{\mathscr{U}}$. Let us define

$$\mathscr{F}(\mathsf{A}, X) := (A, X),$$

which is an object in SA_k.

We shall construct a morphism $\mathscr{F}(\mathsf{f})$ in SA_k, which is the image under $\mathscr{F}$ of a morphism f in $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$, such that

$$\mathscr{F}(\mathsf{g} \circ \mathsf{f}) = \mathscr{F}(\mathsf{g}) \circ \mathscr{F}(\mathsf{f}). \tag{7}$$

Consider a morphism $\mathsf{f} : (\mathsf{A}, X) \to (\mathsf{B}, Y)$, written $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ for short, in SDef_k $(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$. Then, we can cover $X \times_k k((t))$ and $Y \times_k k((t)))$ by Zariski open affine $k((t))$-subvarieties $\mathscr{U}$ and $\mathscr{U}'$, respectively, such that $\mathsf{A} \cap h_{\mathscr{U}}$ and $\mathsf{B} \cap h_{\mathscr{U}'}$ are defined by formulas, $\psi_{\mathscr{U}}$ and $\phi_{\mathscr{U}'}$, respectively, which are the disjunction of formulas of the form (5). We use the covering $\{\mathscr{U}\}$ of $X \times_k k((t))$ and the formula $\psi_{\mathscr{U}}$ to construct a constructible subset $\mathscr{A}$ of $X \times_k k((t))$. In the same way, we use the covering $\{\mathscr{U}'\}$ of $Y \times_k k((t))$ and the formulas $\phi_{\mathscr{U}'}$, to construct a constructible subset $\mathscr{B}$ of $Y \times_k k((t))$. The morphism f induces a morphism of constructible sets $\mathscr{A} \to \mathscr{B}$; hence, for the same reason as the existence of U in (6), the morphism $\mathscr{A} \to \mathscr{B}$ in its turn induces a semi-algebraic morphism of semi-algebraic sets $f : (A, X) \to (B, Y)$. So we define

$$\mathscr{F}(\mathsf{f}) := f,$$

which is well defined and a morphism in SA_k.

Since any morphism f can be factorized through the inclusion into its graph followed by a projection, to check the preserving property (7), it suffices to check for inclusions and projections of small definable T_{acl}-subassignments. By definition, for the B-projection

$$\mathsf{pr}_{\mathsf{B}} : (\mathsf{A} \times \mathsf{B}, X \times_k Y) \to (\mathsf{B}, Y)$$

we have

$$\mathscr{F}(\mathsf{pr}_{\mathsf{B}}) = \mathrm{pr}_B : A \times B \to B,$$

the B-projection of a semi-algebraic subset $A \times B$ of $\mathcal{L}(X \times_k Y) = \mathcal{L}(X) \times_k \mathcal{L}(Y)$ onto B, which is a morphism in SA_k. Also, for a morphism $\mathsf{i}_{\mathsf{AB}} : (\mathsf{A}, X) \to (\mathsf{B}, X)$ in $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ induced by an inclusion $\mathsf{A} \hookrightarrow \mathsf{B}$ of small definable T_{acl}-subassignments, we have $\mathscr{F}(\mathsf{i}_{\mathsf{AB}})(x) = x$ for all $x \in A$. Now, let us consider

$$\mathsf{i} : (\mathsf{C}, X \times_k Y) \to (\mathsf{A} \times \mathsf{B}, X \times_k Y),$$

which is a morphism in the category $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ induced by an inclusion $\mathsf{C} \hookrightarrow \mathsf{A} \times \mathsf{B}$. By definition, it is clear that

$$\mathscr{F}(\mathsf{pr}_{\mathsf{B}} \circ \mathsf{i}) = \mathscr{F}(\mathsf{pr}_{\mathsf{B}}) \circ \mathscr{F}(\mathsf{i}).$$

In the same way, for a morphism

$$\mathsf{j} : (\mathsf{B}, Y) \to (\mathsf{E}, Y)$$

in $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k))$ induced by an inclusion $\mathsf{B} \hookrightarrow \mathsf{E}$, we have

$$\mathscr{F}(\mathsf{j} \circ \mathsf{pr}_{\mathsf{B}}) = \mathscr{F}(\mathsf{j}) \circ \mathscr{F}(\mathsf{pr}_{\mathsf{B}}).$$

We now construct a functor $\mathscr{G}$ from SA_k to $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ which is naturally inverse to $\mathscr{F}$. Let (A, X) be an object of SA_k, i.e., A is a semi-algebraic subset of $\mathcal{L}(X)$. By definition, there exist a cover of X by Zariski open affine k-subvarieties V (viewed as a closed k-subvariety of $\mathbb{A}_k^m$), and for each V, regular functions h_i on V, $1 \leq i \leq m$, a semi-algebraic condition φ (with h_i and φ depending on V) such that

$$A \cap \mathcal{L}(V) = \{x \in \mathcal{L}(V) \mid \varphi(h_1(\tilde{x}), \ldots, h_m(\tilde{x}), \alpha)\}.$$

Clearly,

$$\mathscr{V} := V \times_k \mathrm{Spec} k((t))$$

is embedded over $k[[t]]$ into $\mathbb{A}^m_{k((t))}$, and they form a cover of $X \times_k \mathrm{Spec} k((t))$. Note that φ is a formula in the language $\mathscr{L}_{\mathrm{DP,P}}(k)$, and that each h_i induces a definable morphism of definable subassignments

$$x_i : h_{\mathscr{V}} \to h[1,0,0].$$

Put

$$\mathsf{A}_{\mathscr{V}} = \{x \in h_{\mathscr{V}} \mid \varphi(x_1(x), \ldots, x_m(x), \alpha), \mathrm{ord}_t x_i(x) \geq 0, 1 \leq i \leq m\},$$

and glue all $\mathsf{A}_{\mathscr{V}}$ along the cover $\{\mathscr{V}\}$ of $X \times_k \mathrm{Spec} k((t))$ to get a small definable T_{acl}-subassignment A of $h_{X \times_k \mathrm{Spec} k((t))}$. We can prove that the construction of A is up to definable isomorphism independent of the choice of the cover $\{V\}$. So we can define

$$\mathscr{G}(A, X) = (\mathsf{A}, X),$$

which is an object of $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$. Similarly, we can define $\mathscr{G}(f)$ to be a morphism of $\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ when f is a morphism of SA_k, which satisfies

$$\mathscr{G}(f \circ g) = \mathscr{G}(f) \circ \mathscr{G}(g).$$

The existence of natural isomorphisms

$$\varepsilon : \mathscr{F} \circ \mathscr{G} \to \mathrm{Id}_{\mathrm{SA}_k}$$

and

$$\eta : \mathrm{Id}_{\mathrm{SDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})} \to \mathscr{G} \circ \mathscr{F}$$

follows from the fact that the construction of A from A and vice versa is independent of the choice of covers by open affine subvarieties. □

Let S be an object in $\mathrm{GDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$. Let $\mathrm{RDef}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ denote the full subcategory of $\mathrm{GDef}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ such that each object $\mathsf{X} \to \mathsf{S}$ of $\mathrm{RDef}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ is the S-projection of a definable subassignment X of $\mathsf{S} \times h_{\mathbb{A}_k^n}$, for some n in $\mathbb{N}$. We first mention a special case when $\mathsf{S} = h_S$ as follows. Let S be a closed k-subvariety of $\mathbb{A}_k^d$, for a given d in $\mathbb{N}$. Then the category $\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ defined previously is just the full subcategory of $\mathrm{Def}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$, and its objects $\mathsf{X} \to h_S$ are the h_S-projection of definable subassignments X of $h_{S\times_k \mathbb{A}_k^n}$, with n being variable in $\mathbb{N}$.

Theorem 4.2. *For any algebraic k-variety S, the categories* $\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* Cons_S *are equivalent.*

Proof.

Using the argument in Cluckers and Loeser (2008, Section 16.2), under some elimination theorems, we have that objects of $\mathrm{Def}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ are defined by formulas without quantifiers in the Denef-Pas language $\mathscr{L}_{\mathrm{DP,P}}$. Since Chevalley's constructibility theorem in algebraic geometry (over k) is nothing other than the quantifier elimination theorem for the theory of algebraically closed fields containing k, a formula defining a definable subassignment of $h_{S\times_k \mathbb{A}_k^n}$ defines a constructible subset of $S \times_k \mathbb{A}_k^n$, and via a graph, a definable morphism of definable subassignments gives rise to a constructible morphism of constructible sets, and vice versa. For a detailed argument, we can use the strategy in the proof of Theorem 4.1. □

4.4. Actions

Let X be an algebraic k-variety, and G an algebraic group over k. A *G-action* (or *h_G-action*) on h_X is a definable morphism of definable subassignments

$$h_{G\times_k X} \to h_X$$

such that the corresponding morphism of k-varieties

$$G \times_k X \to X$$

is a G-action on X. The G-action on h_X is called *good* if the corresponding G-action on X is good. In this setting, a definable morphism of definable subassignments

$$h_X \to h_Y$$

is *G-equivariant* if the corresponding morphism of k-varieties

$$X \to Y$$

is G-equivariant. By Theorem 4.2, we can extend this definition of good G-action to that on any definable subassignment of $h[0,n,0]$, for n in $\mathbb{N}$. Let S be a closed k-subvariety of $\mathbb{A}_k^d$, and let S be endowed with a good G-action. Denote by $\mathrm{RDef}_{h_S}^G(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ the subcategory of $\mathrm{GDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ whose objects are G-equivariant definable T_{acl}-morphisms of definable T_{acl}-subassignments $\mathsf{X} \to h_S$, where X is a definable T_{acl}-subassignment of $h_{S\times_k \mathbb{A}_k^n}$, for some n in $\mathbb{N}$, and X is endowed with a good G-action. A morphism in $\mathrm{RDef}_{h_S}^G(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ from an object $\mathsf{X} \to h_S$ to another one $\mathsf{Y} \to h_S$ is a G-equivariant definable T_{acl}-morphism $\mathsf{X} \to \mathsf{Y}$ that commutes with the G-equivariant morphisms to h_S.

Lemma 4.3. *For any k-variety S,* $\mathrm{RDef}_{h_S}^G(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ *and* Cons_S^G *are equivalent.*

Proof.

The lemma is deduced directly from Theorem 4.2 and the definition of good G-action on definable subassignments. □

For an algebraic $k((t))$-variety $\mathscr{X}$, the definable subassignment $h_{\mathscr{X}}$ admits a natural μ_n-action $h_{\mu_n} \times h_{\mathscr{X}} \to h_{\mathscr{X}}$ induced by

$$(\lambda, t) \mapsto \lambda t,$$

for all n in $\mathbb{N}^*$. More precisely, for every K in Field_k, λ in $\mu_n(K)$, and $\varphi(t)$ in $\mathscr{X}(K((t)))$, we have

$$\lambda \cdot \varphi(t) = \varphi(\lambda t).$$

The profinite group scheme $\hat{\mu}$ acts naturally on $h_{\mathscr{X}}$ via μ_n for some n in $\mathbb{N}^*$.

4.5. Grothendieck semirings and rings of definable subassignments

Let S be a definable subassignment. According to Cluckers and Loeser (2008), the Grothendieck semigroup $SK_0(\mathrm{RDef}_{\mathsf{S}})$ of the category $\mathrm{RDef}_{\mathsf{S}}$ is the quotient of the free abelian semigroup generated by symbols $[\mathsf{X} \to \mathsf{S}]$ with $\mathsf{X} \to \mathsf{S}$ being objects in $\mathrm{RDef}_{\mathsf{S}}$ modulo the following relations:

$$[\emptyset \to \mathsf{S}] = 0,$$

$$[\mathsf{X} \to \mathsf{S}] = [\mathsf{Y} \to \mathsf{S}]$$

if $\mathsf{X} \to \mathsf{S}$ and $\mathsf{Y} \to \mathsf{S}$ are isomorphic in $\mathrm{RDef}_{\mathsf{S}}$, and

$$[\mathsf{X} \cup \mathsf{Y} \to \mathsf{S}] + [\mathsf{X} \cap \mathsf{Y} \to \mathsf{S}] = [\mathsf{X} \to \mathsf{S}] + [\mathsf{Y} \to \mathsf{S}]$$

for definable subassignments X and Y of $\mathsf{S} \times h_{\mathbb{A}_k^n}$, for some n in $\mathbb{N}$, and morphisms of X and Y to S factorizing through S-projection. Denote by $K_0(\mathrm{RDef}_{\mathsf{S}})$ the group associated to the Grothendieck semigroup $SK_0(\mathrm{RDef}_{\mathsf{S}})$. If we provide $SK_0(\mathrm{RDef}_{\mathsf{S}})$ and $K_0(\mathrm{RDef}_{\mathsf{S}})$ with a product induced by the fiber product over S of morphisms of subassignments to S defined in Section 2.2 of Cluckers and Loeser (2008), then $SK_0(\mathrm{RDef}_{\mathsf{S}})$ and $K_0(\mathrm{RDef}_{\mathsf{S}})$

are a commutative semiring and ring with unity, respectively. Note that the canonical morphism

$$SK_0(\mathrm{RDef}_{\mathsf{S}}) \to K_0(\mathrm{RDef}_{\mathsf{S}})$$

is not necessarily injective.

Let S be a k-variety endowed with a given G-action. The *G-equivariant Grothendieck group $K_0^G(\mathrm{RDef}_{h_S})$* is the quotient of the free abelian group generated by symbols

$$[\mathsf{X} \to h_S, \sigma]$$

with X being a definable subassignment of $h_{S\times_k \mathbb{A}_k^n}$, for some n in $\mathbb{N}$, endowed with a good G-action σ, and $\mathsf{X} \to h_S$ being a morphism in Def_k, modulo the following relations

$$[\mathsf{X} \to h_S, \sigma] = [\mathsf{Y} \to h_S, \sigma']$$

if there exists a G-equivariant definable morphism $\mathsf{X} \to \mathsf{Y}$ that commutes with the definable morphisms to h_S,

$$[\mathsf{X} \to h_S, \sigma] = [\mathsf{Y} \to h_S, \sigma|_{\mathsf{Y}}] + [\mathsf{X}\setminus\mathsf{Y} \to h_S, \sigma|_{\mathsf{X}\setminus\mathsf{Y}}]$$

for Y being a σ-stable definable subassignment of X, and

$$[\mathsf{X}\times h_{\mathbb{A}_k^m} \to h_S, \sigma] = [\mathsf{X}\times h_{\mathbb{A}_k^m} \to h_S, \sigma']$$

if σ and σ' lift the same G-action on X to an affine action on $\mathsf{X}\times h_{\mathbb{A}_k^m}$, for any $m \geq 0$. As above, with respect to the fiber product of subassignments endowed with diagonal G-action, the group $K_0^G(\mathrm{RDef}_{h_S})$ is a commutative ring with unity.

The Grothendieck rings $K_0(\mathrm{RDef}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}}))$ and $K_0(\mathrm{RDef}_{h_S}^G(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}}))$ of the categories $\mathrm{RDef}_{\mathsf{S}}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$ and $\mathrm{RDef}_{h_S}^G(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})$, respectively, are defined in the same way as before.

Lemma 4.4. *For any k-variety S, there are canonical isomorphisms*

$$K_0(\mathrm{RDef}_{h_S}(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \cong K_0(\mathrm{Var}_S)$$

and

$$K_0(\mathrm{RDef}_{h_S}^G(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \cong K_0^G(\mathrm{Var}_S).$$

Proof.

This statement is a direct corollary of Theorem 4.2 and Lemma 4.3. We can also refer to Cluckers and Loeser (2008, Section 16.2) for a proof of the first isomorphism. □

5. INTEGRABLE FUNCTIONS AND MEASURABLE SUBASSIGNMENTS

5.1. Rings of motivic functions and Functions

Let S be a definable subassignment. Put

$$\mathbb{A} := \mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, \left(\frac{1}{1-\mathbb{L}^{-n}}\right)_{n\in\mathbb{N}^*}\right],$$

where, by abuse of notation, $\mathbb{L}$ also stands for the class of $\mathsf{S} \times h_{\mathbb{A}^1_k}$ in $K_0(\mathrm{RDef}_{\mathsf{S}})$. By Cluckers and Loeser (2008), for any real number $q > 1$, there is a unique morphism of rings

$$v_q : \mathbb{A} \to \mathbb{R}$$

sending $\mathbb{L}$ to q, and such that, whenever q is transcendental, v_q is injective. Denote by $\mathbb{A}_+$ the subset of $\mathbb{A}$ consisting of elements a with $v_q(a) \geq 0$.

We now recall Section 4.6 of Cluckers and Loeser (2008). Denote by $|\mathsf{S}|$ the set of points of S. Let $\mathcal{P}(\mathsf{S})$ be the subring of the ring of functions $|\mathsf{S}| \to \mathbb{A}$ which is generated by constant functions

$$|\mathsf{S}| \to \mathbb{A},$$

by functions

$$\tilde{\alpha} : |\mathsf{S}| \to \mathbb{Z},$$

and by functions

$$\mathbb{L}^{\tilde{\beta}} : |\mathsf{S}| \to \mathbb{A},$$

for definable morphisms $\alpha, \beta : \mathsf{S} \to h_{\mathbb{Z}} = h[0,0,1]$. Here, notice that to any definable morphism

$$\alpha : \mathsf{S} \to h[0,0,1]$$

corresponds a function

$$\tilde{\alpha} : |\mathsf{S}| \to \mathbb{Z}.$$

Denote by $\mathcal{P}_+(\mathsf{S})$ the semiring of functions in $\mathcal{P}(\mathsf{S})$ with values in $\mathbb{A}_+$. In particular, the ring $\mathcal{P}(h_{\mathrm{Spec}k})$ and the semiring $\mathcal{P}_+(h_{\mathrm{Spec}k})$ are nothing but $\mathbb{A}$ and $\mathbb{A}_+$, respectively. Denote by $\mathcal{P}^0(\mathsf{S})$ the subring of $\mathcal{P}(\mathsf{S})$, which is generated by $\mathbb{L}-1$ and by character functions $\mathbf{1}_{\mathsf{X}}$ for all definable subassignments X of S, and also define

$$\mathcal{P}^0_+(\mathsf{S}) := \mathcal{P}^0(\mathsf{S}) \cap \mathcal{P}_+(\mathsf{S}).$$

According to Cluckers and Loeser (2008, Section 5.3), the semiring $\mathcal{C}_+(\mathsf{S})$ of *positive constructible motivic functions* on S and the ring $\mathcal{C}(\mathsf{S})$ of *constructible motivic functions* on S are defined as follows

$$\begin{aligned}\mathcal{C}_+(\mathsf{S}) &:= SK_0(\mathrm{RDef}_{\mathsf{S}}) \otimes_{\mathcal{P}^0_+(\mathsf{S})} \mathcal{P}_+(\mathsf{S}),\\ \mathcal{C}(\mathsf{S}) &:= K_0(\mathrm{RDef}_{\mathsf{S}}) \otimes_{\mathcal{P}^0(\mathsf{S})} \mathcal{P}(\mathsf{S}).\end{aligned} \tag{8}$$

If S is an algebraic k-variety endowed with a good $\hat{\mu}$-action, we define

$$\mathcal{C}^{\hat{\mu}}(h_S) := K_0^{\hat{\mu}}(\mathrm{RDef}_{h_S}) \otimes_{\mathcal{P}^0(h_S)} \mathcal{P}(h_S).$$

As mentioned in Section 16.1 of Cluckers and Loeser (2008), with respect to the language $\mathscr{L}_{\mathrm{DP,P}}(k)$ and theory T_{acl}, we can define rings $\mathcal{P}_+(\mathsf{S},(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$, $\mathcal{P}(\mathsf{S},(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$, $\mathcal{C}_+(\mathsf{S},(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$, and in the same way, the ring $\mathcal{C}^{\hat{\mu}}(h_S,(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$.

In the rest of the present paper, we shall not work with the rings $\mathcal{C}(h_S)$, $\mathcal{C}(h_S,(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$, and $\mathcal{C}^{\hat{\mu}}(h_S,(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$ except the trivial case $S=\mathrm{Spec}k$. For this trivial case, we have the following lemma, which is obvious from the definition.

Lemma 5.1. *There exist canonical isomorphisms*

$$\begin{aligned}\mathcal{C}_+(h_{\mathrm{Spec}k}) &\cong SK_0(\mathrm{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} \mathbb{A}_+,\\ \mathcal{C}(h_{\mathrm{Spec}k},(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})) &\cong \mathcal{M}_{\mathrm{loc}},\\ \mathcal{C}^{\hat{\mu}}(h_{\mathrm{Spec}k},(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}})) &\cong \mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}.\end{aligned}$$

The important properties of $\mathcal{C}_+(\mathsf{S})$ and $\mathcal{C}(\mathsf{S})$ are given in Section 5 of Cluckers and Loeser (2008).

According to Cluckers and Loeser (2008, Section 3), the K-dimension of a definable subassignment (with K a capital letter not a mathematical notation) is defined as follows. If S is a definable subassignment of $h_{\mathscr{X}}$, with $\mathscr{X}$ being an algebraic $k((t))$-variety, then the K-*dimension* of S, denoted by $\mathrm{Kdim}\mathsf{S}$, is the dimension of the $k((t))$-variety which is the intersection of all $k((t))$-subvarieties $\mathscr{Y}$ of $\mathscr{X}$ with $h_{\mathscr{Y}}$ containing S. It may happen that the intersection is empty; in that case, we define

$$\mathrm{Kdim}\mathsf{S} := -\infty.$$

If S is a definable subassignment of $h_{\mathscr{X}\times X\times\mathbb{Z}^r}$ with $\mathscr{X}$ as above and X as k-variety, then we define

$$\mathrm{Kdim}\mathsf{S} := \mathrm{Kdim}\mathsf{pr}_1(\mathsf{S}),$$

where

$$\mathsf{pr}_1 : h_{\mathscr{X}\times X\times\mathbb{Z}^r} \to h_{\mathscr{X}}$$

is the first projection.

A positive constructible motivic function φ in $\mathcal{C}_+(\mathsf{S})$ is called of K-dimension $\le d$ if φ is a finite sum $\sum_i \alpha_i \mathbf{1}_{\mathsf{S}_i}$ in $\mathcal{C}_+(\mathsf{S})$ such that the K-dimension of every S_i is $\le d$. Let $\mathcal{C}_+^{\le d}(\mathsf{S})$ be the sub-semigroup of $\mathcal{C}_+(\mathsf{S})$ of elements of K-dimension $\le d$, and

$$C_+^d(\mathsf{S}) := \mathcal{C}_+^{\le d}(\mathsf{S})/\mathcal{C}_+^{\le d-1}(\mathsf{S})$$

and

$$C_+(\mathsf{S}) := \bigoplus_{d\geq 0} C_+^d(\mathsf{S}).$$

An element in $C_+(\mathsf{S})$ is called a *positive constructible motivic Function* on S (with the capital letter F). Clearly, $C_+(\mathsf{S})$ is a graded abelian semigroup and has a module structure over the semiring $\mathcal{C}_+(\mathsf{S})$ (see Cluckers & Loeser, 2008, Section 6).

5.2. Integrable positive Functions and measurable subassignments

Let S be in Def_k. By Cluckers and Loeser (2008, Theorem 10.1.1), there exists a unique functor $\mathrm{I}_{\mathsf{S}}C_+$ from the category $\mathrm{Def}_{\mathsf{S}}$ to the category of abelian semigroups that sends every morphism

$$\mathsf{f} : \mathsf{X} \to \mathsf{Y}$$

in $\mathrm{Def}_{\mathsf{S}}$ to a morphism of semigroups

$$\mathsf{f}_! : \mathrm{I}_{\mathsf{S}}C_+(\mathsf{X}) \to \mathrm{I}_{\mathsf{S}}C_+(\mathsf{Y})$$

and satisfies axioms A0–A8. If $\mathsf{S} = h_{\mathrm{Spec}k}$ we write $\mathrm{I}C_+(\mathsf{X})$ instead of $\mathrm{I}_{\mathsf{S}}C_+(\mathsf{X})$, and call it the semigroup of integrable positive Functions on X. By Proposition 12.2.2 of Cluckers and Loeser (2008), if X is a definable subassignment of $h[m,n,0]$ that is *bounded*, i.e., there exists an $s \in \mathbb{N}$ such that X is contained in the subassignment of $h[m,n,0]$ defined by

$$\mathrm{ord}_t x_i \geq -s$$

for all $1 \leq i \leq m$, then $[\mathbf{1}_{\mathsf{X}}]$ belongs to $\mathrm{I}C_+(\mathsf{X})$, where $\mathbf{1}_{\mathsf{X}}$ is the characteristic function on X. (In the previous definition of boundedness, if $s = 0$ then X is said to be *positively bounded*.)

Also, in the trivial case $\mathsf{S} = h_{\mathrm{Spec}k}$, let us take f to be the projection of X onto the final subassignment $h_{\mathrm{Spec}k}$ of Def_k. We denote by $\widetilde{\mu}$ the morphism of semigroups $\mathsf{f}_!$, namely,

$$\widetilde{\mu} : \mathrm{I}C_+(\mathsf{X}) \to \mathrm{I}C_+(h_{\mathrm{Spec}k}) \cong \mathcal{C}_+(h_{\mathrm{Spec}k}).$$

Applying Section 16.1 of Cluckers and Loeser (2008), we have a canonical morphism of rings

$$\mathcal{C}_+(h_{\mathrm{Spec}k}) \to \mathcal{C}_+(h_{\mathrm{Spec}k}, (\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})).$$

On the other hand, we also have another canonical morphism

$$\mathcal{C}_+(h_{\mathrm{Spec}k}, (\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \to \mathcal{C}(h_{\mathrm{Spec}k}, (\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}})) \cong \mathcal{M}_{\mathrm{loc}}.$$

Taking the composition of the last two morphisms with $\widetilde{\mu}$, we get a morphism of rings

$$\mu : \mathrm{I}C_+(\mathsf{X}) \to \mathcal{M}_{\mathrm{loc}}.$$

As mentioned previously, if X is a bounded definable subassignment in Def_k, then $[\mathbf{1}_{\mathsf{X}}]$ is in $\mathrm{I}C_+(\mathsf{X})$. In this case, we call X a *motivically measurable* (definable) subaasignment. We define the *motivic measure* of X to be

$$\mu(\mathsf{X}) := \mu([\mathbf{1}_{\mathsf{X}}]),$$

which lies in $\mathcal{M}_{\mathrm{loc}}$. By the additivity of the integral (Axiom A2) in Cluckers and Loeser (2008, Theorem 10.1.1), the motivic measure μ is additive on bounded definable subassignments.

Denote by $\widehat{\mathcal{M}}_k$ the completion of $\mathcal{M}_k$ in the sense of Denef and Loeser (1999), and by δ the canonical morphism $\mathcal{M}_{\mathrm{loc}} \to \widehat{\mathcal{M}}_k$ defined by the expansion of $1-\mathbb{L}^{-n}$, for every n in $\mathbb{N}^*$.

Proposition 5.2. *Let X be an algebraic k-variety, A a semi-algebraic subset of $\mathcal{L}(X)$, and* A *the small definable subassignment corresponding to A via the equivalence of categories between* $\mathrm{SA}_k(X)$ *and* $\mathrm{SDef}_k(X,(\mathscr{L}_{\mathrm{DP,P}}(k),T_{\mathrm{acl}}))$ *in Theorem 4.1. Then*

$$\delta(\mu(\mathsf{A})) = \mu'(A),$$

where μ' is Denef-Loeser's motivic volume defined in Denef and Loeser (1999).

Proof.
Note that if a definable function

$$\alpha : \mathsf{A} \to h_{\mathbb{Z}}$$

in the language $\mathscr{L}_{\mathrm{DP,P}}(k)$ is the zero function on A, then the semi-algebraic function

$$\tilde{\alpha} : A \to \mathbb{Z}$$

corresponding to α via the equivalence of categories of $\mathrm{SA}_k(X)$ and $\mathrm{SDef}_k(X,(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}}))$ in Theorem 4.1 is also the zero function on A. Now applying Theorem 16.3.1 of Cluckers and Loeser (2008) to $\alpha = 0$, we get Proposition 5.2. □

5.3. Invariant definable subassignments and their measure

Let m,n be in $\mathbb{N}$, and $\gamma = (\gamma_1,\ldots,\gamma_m)$ be in $\mathbb{Z}^m$. A definable subassignment X of $h[m,n,0]$ is called *γ-invariant* if, for every K in Field_k, (a,b) and (x,y) in

$$h[m,n,0](K) = K((t))^m \times K^n$$

satisfying

$$\mathrm{ord}_t x_i \geq \gamma_i$$

for $1 \leq i \leq m$, both elements (a,b) and $(a,b)+(x,y)$ are simultaneously in either $\mathsf{X}(K)$ or in the complement of $\mathsf{X}(K)$ in $K((t))^m \times K^n$. A definable subassignment of $h[m,n,0]$ is called *invariant* if it is γ-invariant for some γ in $\mathbb{Z}^m$. In the case $\gamma_i = \beta \in \mathbb{Z}$ for all $1 \leq i \leq m$, we write β-invariant instead of γ-invariant. Note that if X is γ-invariant and $\gamma_i \leq \gamma_i'$ for all $1 \leq i \leq m$, then X is also $(\gamma_1',\ldots,\gamma_m')$-invariant. It is a fact that any bounded definable subassignment of $h[m,0,0]$ closed in the valuation topology is γ-invariant for some γ in $\mathbb{Z}^m$.

Now, let β be in $\mathbb{N}^*$, and let X be a bounded definable subassignment of $h[m,n,0]$ with $\mathrm{ord}_t x_i \geq 0$ for every $x = (x_1, \ldots, x_m)$ on X and for all $1 \leq i \leq m$. Then X is β-invariant if and only if there exists a constructible subset X_β of

$$\mathbb{A}_k^{\beta m} \times_k \mathbb{A}_k^n \cong \mathcal{L}_{\beta-1}(\mathbb{A}_k^m) \times \mathbb{A}_k^n$$

such that, for every K in Field_k,

$$\mathsf{X}(K) = \left(\pi_\beta(K)\right)^{-1}\left(X_\beta(K)\right),$$

which is the pullback of $X_\beta(K)$ under the canonical map

$$\pi_\beta(K) : K[[t]]^m \times K^n \to \left(K[t]/(t^\beta)\right)^m \times K^n \cong K^{\beta m} \times K^n.$$

Indeed, we can show this by observing that, for K in Field_k, every fiber of the restriction $\pi_\beta(K)$ is definably bijective to the set $(t^\beta)K[[t]]^m$. The maps $\pi_\beta(K)$ induce the canonical morphism

$$\pi_\beta : h[m,n,0] \to h[0, \beta m + n, 0],$$

and X is β-invariant if and only if there exists a constructible subset $X_\beta \subseteq \mathbb{A}_k^{\beta m} \times_k \mathbb{A}_k^n$ such that $\mathsf{X} = \pi_\beta^{-1}(h_{X_\beta})$. By abuse of notation, we shall denote the canonical morphism $\mathsf{X} \to h_{X_\beta}$ by π_β.

Lemma 5.3. *Let $\beta \leq \beta'$ be in $\mathbb{N}^*$, and let X and X_β be as before. Then the identity*

$$[h_{X_{\beta'}}] = [h_{X_\beta}]\mathbb{L}^{(\beta'-\beta)m}$$

holds in $K_0(\mathrm{RDef}_k)$. As a consequence, the element $[h_{X_\beta}]\mathbb{L}^{-(\beta+1)m}$ in the ring $K_0(\mathrm{RDef}_k)[\mathbb{L}^{-1}]$ is independent of the choice of sufficiently large β.

Proof.

The natural map

$$\mathcal{L}_{\beta'-1}(\mathbb{A}_k^m) \times_k \mathbb{A}_k^n \to \mathcal{L}_{\beta-1}(\mathbb{A}_k^m) \times_k \mathbb{A}_k^n$$

induced by truncation is a Zariski locally trivial fibration with fiber $\mathbb{A}_k^{(\beta'-\beta)m}$. Along this map, $X_{\beta'}$ is the preimage of X_β. Thus, we get the identity $[h_{X_{\beta'}}] = [h_{X_\beta}]\mathbb{L}^{(\beta'-\beta)m}$ in $K_0(\mathrm{RDef}_k)$. □

We denote by $\mathrm{vol}(\mathsf{X})$ the image of $[h_{X_\beta}]\mathbb{L}^{-(\beta+1)m} \in K_0(\mathrm{RDef}_k)[\mathbb{L}^{-1}]$ under the canonical morphism (due to Cluckers and Loeser (2008, Section 16.1))

$$K_0(\mathrm{RDef}_k)[\mathbb{L}^{-1}] \to K_0(\mathrm{RDef}_k(\mathscr{L}_{\mathrm{DP,P}}(k), T_{\mathrm{acl}}))[\mathbb{L}^{-1}] \cong \mathcal{M}_k.$$

Theorem 5.4. *Let X be an invariant bounded definable subassignment of $h[m,n,0]$ such that, for every (x,y) on X with $x = (x_1, \ldots, x_m)$, $\mathrm{ord}_t x_i \geq 0$ for all $1 \leq i \leq m$. Then the identity*

$$\mu(\mathsf{X}) = \mathrm{loc}(\mathrm{vol}(\mathsf{X}))$$

holds in $\mathcal{M}_{\mathrm{loc}}$.

Proof.

Assume that X is β-invariant, for some β in $\mathbb{N}^*$. For $k((t))$-coordinates $(x_1,\ldots,x_m)$ in X, let us write

$$x_i = a_{i0} + a_{i1}t + \cdots + a_{i,\beta-1}t^{\beta-1} + \cdots, \quad 1 \leq i \leq m.$$

Consider the inclusion

$$\mathsf{i} : h[m,n,0] \hookrightarrow h[m,\beta m+n,0]$$

given by $(x,y) \mapsto (x,(a_{i0},a_{i1},\ldots,a_{i,\beta-1})_{1\leq i\leq m},y)$, and the projection

$$\mathsf{pr} : h[m,\beta m+n,0] \to h[0,\beta m+n,0]$$

given by $(x,z) \mapsto z$. Denote by $\mathsf{i_X}$ the restriction of i on X. We can regard $\mathsf{i_X}$ as an inclusion

$$\mathsf{i_X} : \mathsf{X} \to \mathsf{X}[0,\beta m,0].$$

Denote by $\mathsf{pr_X}$ the restriction of pr on $\mathsf{X}[0,\beta m,0]$. Then the composition $\mathsf{pr_X} \circ \mathsf{i_X}$ is nothing but the canonical map $\pi_\beta : \mathsf{X} \to h_{X_\beta}$. By the functoriality (Axiom A0) of the integral in Cluckers and Loeser (2008, Theorem 10.1.1), we have $(\pi_\beta)_! = (\mathsf{pr_X})_! \circ (\mathsf{i_X})_!$, hence,

$$(\pi_\beta)_!\big([\mathbf{1}_\mathsf{X}]\big) = (\mathsf{pr_X})_!\big([\mathbf{1}_{\mathsf{i(X)}}]\big).$$

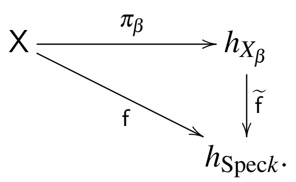
Since X is β-invariant, by fixing an element (a,b) in $\mathsf{i(X)}$, we have

$$\mathsf{i(X)} = \mathsf{pr}_\mathsf{X}^{-1}(h_{X_\beta}) = \{(a,b)+(x,y) \in h[m,\beta m+n,0] \mid \mathrm{ord}_t x_i \geq \beta \ \forall\, 1 \leq i \leq m\}.$$

By definition, constructible motivic Functions on $\mathsf{i(X)}$ are equivalence classes of elements of $C_+(\mathsf{i(X)})$ modulo support of smaller dimension dimension (see Cluckers & Loeser, 2005, Section 3.3; Cluckers & Loeser, 2008, Section 6). Hence, in $\mathrm{I}C_+(\mathsf{i(X)})$, we have $\big[\mathbf{1}_{\mathsf{i(X)}}\big] = \big[\mathbf{1}_{\widetilde{\mathsf{X}}}\big]$, where $\widetilde{\mathsf{X}}$ is defined similar to $\mathsf{i(X)}$ with $\mathrm{ord}_t x_i = \beta$ replacing $\mathrm{ord}_t x_i \geq \beta$. Now, applying Axiom A7 of Cluckers and Loeser (2008, Theorem 10.1.1) inductively, we get

$$(\mathsf{pr_X})_!\big([\mathbf{1}_{\mathsf{i(X)}}]\big) = \mathbb{L}^{-(\beta+1)m}\big[\mathbf{1}_{h_{X_\beta}}\big].$$

The projection f of X onto the final object $h_{\mathrm{Spec}k}$ in Def_k can be factorized through the canonical map $\pi_\beta : \mathsf{X} \to h_{X_\beta}$. Thus, we have the following commutative diagram



Therefore,

$$\mu(\mathsf{X}) = \mathsf{f}_!([\mathbf{1}_\mathsf{X}]) = \widetilde{\mathsf{f}}\big(\mathbb{L}^{-(\beta+1)m}\big[\mathbf{1}_{h_{X_\beta}}\big]\big).$$

By Cluckers and Loeser (2008, Proposition 5.3.1), we have

$$\begin{aligned}\mathcal{C}_+(h[0,\beta m,0]) &\cong SK_0(\mathrm{RDef}_{h[0,\beta m,0]})\otimes_{\mathcal{P}^0_+(h_{\mathrm{Spec}k})}\mathcal{P}_+(h_{\mathrm{Spec}k})\\ &\cong SK_0(\mathrm{RDef}_{h[0,\beta m,0]})\otimes_{\mathbb{N}[\mathbb{L}-1]}\mathbb{A}_+.\end{aligned}$$

Thus, element $\left[\mathbf{1}_{h_{X_\beta}}\right]$ in $\mathcal{C}_+(h_{X_\beta})$ can be written as $\left[h_{X_\beta}\to h_{X_\beta}\right]\otimes 1$. Then the identities

$$\mu(\mathsf{X})=\mathbb{L}^{-(\beta+1)m}\left[h_{X_\beta}\to h_{X_\beta}\to h_{\mathrm{Spec}k}\right]=\mathbb{L}^{-(\beta+1)m}\left[h_{X_\beta}\right]=\mathrm{loc}(\mathrm{vol}(\mathsf{X}))$$

hold true in $\mathcal{M}_{\mathrm{loc}}$. □

Let us recall some settings in Section 14.5 of Cluckers and Loeser (2008) and Section 4.3 of Lê and Nguyen (2020) on the ramification. Consider a formula φ in the language $\mathscr{L}_{\mathrm{DP,P}}(k[t])$, i.e., the coefficients of φ are in $k[t]$ in the valued field sort and in k in the residue field sort, such that φ has m free variables in the valued field sort, n free variables in residue field sort, and r free variables in the value group sort. For each e in $\mathbb{N}^*$, let $\varphi[e]$ denote the formula obtained from φ by replacing t everywhere by t^e. If X is a definable subassignment of $h[m,n,r]$ defined by φ, we denote by $\mathsf{X}[e]$ the definable subassignment of $h[m,n,r]$ defined by $\varphi[e]$. In addition, if X is bounded, then so is $\mathsf{X}[e]$, that is, $[\mathbf{1}_{\mathsf{X}[e]}]$ is in $IC_+(\mathsf{X}[e])$, for every e in $\mathbb{N}^*$ (see Cluckers & Loeser, 2008, Proposition 14.5.1).

Proposition 5.5 (Cluckers & Loeser, 2008, Theorem 14.5.3)**.** *Assume that X is a bounded definable subassignment of $h[m,0,0]$ defined by a formula in $\mathscr{L}_{\mathrm{DP,P}}(k[t])$ with m free variables in the valued field sort. Then the formal power series*

$$Z_{\mathsf{X}}(T)=\sum_{e\in\mathbb{N}^*}\mu(\mathsf{X}[e])T^e$$

is in $\mathcal{M}_{\mathrm{loc}}[[T]]_{\mathrm{sr}}$.

Let β be in $\mathbb{N}^*$, and X be a β-invariant bounded definable subassignment of $h[m,n,0]$ defined by a formula in $\mathscr{L}_{\mathrm{DP,P}}(k[t])$. Then, for every e in $\mathbb{N}^*$, the bounded definable subassignment $\mathsf{X}[e]$ of $h[m,n,0]$ is βe-invariant. Therefore, there exists a constructible subset $X_{\beta e}$ of $\mathcal{L}_{\beta e-1}(\mathbb{A}^m_k)\times_k\mathbb{A}^n_k$ such that, for every K in Field_k, $\mathsf{X}[e](K)$ is the pullback of $X_{\beta e}(K)$ under the canonical map

$$K((t))^m\times K^n\to\left(K[t]/(t^{\beta e})\right)^m\times K^n.$$

Proposition 5.6. *Let β be in $\mathbb{N}^*$, and let X be a β-invariant bounded definable subassignment of $h[m,n,0]$ defined by a formula φ in $\mathscr{L}_{\mathrm{DP,P}}(k[t])$ not containing the symbol $\overline{\mathrm{ac}}$. Then the definable subassignment $h_{X_{\beta e}}$ is stable by the natural μ_e-action on $h[m,n,0]$ defined by*

$$\lambda\cdot(x,\xi)=(\lambda x,\xi),\quad \textit{with}\quad \lambda x(t)=x(\lambda t).$$

As a consequence, the quantity $\mathrm{vol}(\mathsf{X}[e])$ is an element in $\mathcal{M}^{\mu_e}_k$, and the identity

$$\mu(\mathsf{X}[e])=\mathrm{loc}(\mathrm{vol}(\mathsf{X}[e]))$$

holds in $\mathcal{M}_{\text{loc}}^{\mu_e}$*; thus, the series*

$$Z_{\mathsf{X}}(T) = \sum_{e \in \mathbb{N}^*} \mu(\mathsf{X}[e]) T^e$$

is in $\mathcal{M}_{\text{loc}}^{\hat{\mu}}[[T]]_{\text{sr}}$.

Proof.

By the definition of β-invariance, we have

$$h_{X_{\beta e}} \cong \{(x,\xi) \in h[m,n,0] \mid \varphi[e](x,\xi), 0 \leq \text{ord}_t x_i \leq \beta e - 1, 1 \leq i \leq m\}.$$

By Corollary 2.1.2 of Cluckers and Loeser (2008), the $\mathscr{L}_{\text{DP,P}}(k[t])$-formula φ is a finite disjunction of formulas of the form

$$\psi(\overline{\text{ac}} g'_1(x), \ldots, \overline{\text{ac}} g'_{l'}(x), \xi) \wedge \vartheta(\text{ord}_t f'_1(x), \ldots, \text{ord}_t f'_l(x), \alpha),$$

where f'_i and g'_j are polynomials over $k[t]$, ψ is an $\mathbf{L}_{\text{Rings}}$-formula, and ϑ is an $\mathbf{L}_{\text{PR}}$-formula. Since φ is equivalent to an $\mathscr{L}_{\text{DP,P}}(k[t])$-formula without angular component symbol $\overline{\text{ac}}$ due to the hypothesis, $\varphi[e]$ is a finite disjunction of formulas of the form

$$\psi_1((g_j(x))_j) \wedge \psi_2(\xi) \wedge \vartheta((\text{ord}_t f_i(x))_i, \alpha),$$

where f_i and g_j are polynomials over $k[t^e]$, ψ_1 and ψ_2 are $\mathbf{L}_{\text{Rings}}$-formulas, and ϑ is an $\mathbf{L}_{\text{PR}}$-formula. If in $f_i(x(t))$ and $g_j(x(t))$ we replace t by λt, then for any $x(t)$ in $K((t))^m$, λ in $\mu_e(K)$, and K in Field_k, we get expressions of $f_i(x(\lambda t))$ and $g_j(x(\lambda t))$, since the coefficients of polynomials f_i and g_j in $k[t^e][x]$ do not change. This proves that if (x,ξ) is in $h_{X_{\beta e}}$, so is $(\lambda x, \xi)$, for any λ in μ_e; that is, $h_{X_{\beta e}}$ is stable under the action of μ_e. □

Now assume that X is small. As mentioned above, there is a canonical action of $\hat{\mu}$ on $h[m,0,0]$ induced by

$$(\lambda, t) \mapsto \lambda t.$$

We say that the definable subassignment X is $\hat{\mu}$-stable if there exists an $n \in \mathbb{N}^*$ such that, for every $x = (x_1(t), \ldots, x_m(t))$ in X and λ in μ_n, the point

$$\lambda \cdot x = (x_1(\lambda t), \ldots, x_m(\lambda t))$$

is in X. Since formulas defining X are in the Denef-Pas language, by quantifier elimination for algebraically closed fields, they also define a semi-algebraic subset X of some arc space $\mathcal{L}(\mathbb{A}_k^m)$ of $\mathbb{A}_k^m$. The assignment

$$\mathsf{X} \mapsto X$$

carries the canonical $\hat{\mu}$-action on $h[m,0,0]$ to the canonical $\hat{\mu}$-action on $\mathcal{L}(\mathbb{A}_k^m)$, and in that way, X is also $\hat{\mu}$-stable in $\mathcal{L}(\mathbb{A}_k^m)$. As in Cluckers and Loeser (2008, Theorem 16.3.1, Remark 16.3.2), we can see that X is measurable as X is measurable, and that with the above action, since $\mu'(X)$ is in $\mathcal{M}_{\text{loc}}^{\hat{\mu}}$, the measure $\mu(\mathsf{X})$ of X is also in $\mathcal{M}_{\text{loc}}^{\hat{\mu}}$. Here, as in Cluckers and Loeser (2008, Theorem 16.3.1), μ' stands for Denef-Loeser's motivic measure (Denef & Loeser, 1999), and further, by Cluckers and Loeser (2008, Remark 16.3.2), we can consider that this measure takes a value in $\mathcal{M}_{\text{loc}}^{\hat{\mu}}$ for the context with $\hat{\mu}$-action.

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