

THE THOM CONDITION

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Abstract

In this note we explain the notion of the Thom condition for the Whitney stratifications of a complex analytic map. We give a question P. Deligne and indicate a possible way to answer it.

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1. INTRODUCTION

Thom (1969) introduced a condition for a stratified map to have a good topological behavior.

Namely, let $f : X \rightarrow Y$ be complex analytic map. We say:

Definition 1.1. *The complex analytic map f can be stratified if there are a complex analytic Whitney stratification $(S_\alpha)_{\alpha \in A}$ of X and a complex analytic Whitney stratification $(T_\beta)_{\beta \in B}$ of Y such that, for each $\alpha \in A$, there is a index $\beta(\alpha) \in B$ such that the map f induces a surjective and submersive map from A_α onto $B_{\beta(\alpha)}$. We call the stratifications (S_α) and (T_β) the stratifications of the stratified map f .*

Now, let f be a complex analytic stratified map by the stratification (S_α) of X and the stratification (T_β) of Y .

Definition 1.2. *We say that the map f satisfies the Thom condition if it is stratified by the Whitney stratifications (S_α) and (T_β) , for any pair of strata $(S_{\alpha_1}, S_{\alpha_2})$ of X such that $S_{\alpha_1} \subset \bar{S}_{\alpha_2}$, for any sequence (x_n) of S_{α_2} of points of X which tends to $x \in S_{\alpha_1}$ and for which the sequence of tangent spaces $T_{x_n}(f_2^{-1}(f_2(x_n)))$ has a limit \mathbb{T} . This limit contains $T_x(f_1^{-1}(f_1(x)))$, where f_i is the restriction of f to S_{α_i} , for $i = 1, 2$.*

Since f is a complex analytic stratified map, the restriction of f to a stratum S_α has a maximal rank. Therefore, the tangent spaces $T_{x_n}(f_2^{-1}f_2(x_n))$ have the same dimension.

Of course, there are complex analytic stratified maps that do not satisfy the Thom condition. For instance, consider the blow-up of the origin in \mathbb{C}^2 :

$$e : E \rightarrow \mathbb{C}^2.$$

We have a stratified map with a Whitney stratification $(E - e^{-1}(0), S_\alpha)$ of E , where (S_α) is a Whitney stratification of $E^{-1}(0)$, and the stratification $\mathbb{C}^2 - \{0\}, \{0\}$ of \mathbb{C}^2 . Since the fibers of points in $\mathbb{C}^2 - \{0\}$ are points, the fibers of e have dimension 0. If (x_n) is a sequence of points of $E - e^{-1}(0)$ that tends to a point x in a stratum S_α of dimension 1, the Thom condition cannot be satisfied. This tells us that the existence of stratifications with the Thom condition is a particular property of complex analytic maps.

2. THE MILNOR FIBRATION

In his book, Milnor (1968) gave a property associated to singularities of complex analytic function.

Namely, let g be a germ of a nonconstant complex analytic function of $(\mathbb{C}^{n+1}, 0)$. We may assume that $g(0) = 0$. We have (see Theorem (4.8) of Milnor (1968)):

Theorem 2.1. (J. Milnor) *There is $\varepsilon_0 > 0$ such that for any $\varepsilon, \varepsilon_0 \geq \varepsilon > 0$ the map:*

$$\varphi_\varepsilon : \mathbb{S}_\varepsilon(0) - \{g = 0\} \rightarrow \mathbb{S}^1,$$

is a smooth locally trivial fibration where $\mathbb{S}_\varepsilon(0)$ is the sphere of \mathbb{C}^{n+1} centered at the point 0 with radius ε , \mathbb{S}^1 is the unit circle of \mathbb{C} centered at 0, and φ_ε is defined by:

$$\varphi_\varepsilon(z) = \frac{g(z)}{|g(z)|}$$

for $z \in \mathbb{S}_\varepsilon(0) - \{g = 0\}$.

A remarkable observation is that Milnor does not assume the point 0 to be an isolated critical point of g . If one assumes that 0 is an isolated critical point, one has a local fibration:

Theorem 2.2. *Suppose that the complex analytic function g has an isolated critical point at 0. There is $\varepsilon_1 > 0$, such that for any $\varepsilon, \varepsilon_1 > \varepsilon > 0$, there is $\eta(\varepsilon)$ such that, for any $\eta, \eta(\varepsilon) > \eta > 0$, the function g induces a locally trivial smooth fibration:*

$$\varphi(\varepsilon, \eta) : \mathbb{B}_\varepsilon(0) \cap g^{-1}(\mathbb{S}_\eta^1) \rightarrow \mathbb{S}_\eta^1$$

where $\mathbb{B}_\varepsilon(0)$ is the closed ball of \mathbb{C}^{n+1} centered at 0 with radius ε and \mathbb{S}_η^1 is the circle centered at 0 with radius η in \mathbb{C} .

Proof.

The proof of Theorem 2.2 will be reminiscent of the condition of Thom. Let η be small enough such that $\varepsilon \gg \eta > 0$. Then the hypersurface $\{g = t\}$, for any $t, \eta > |t| > 0$, is nonsingular inside a neighbourhood V of 0 sufficiently small. We may assume that this neighbourhood of 0 contains $\mathbb{B}_\varepsilon(0)$. \square

Since the critical point 0 is isolated in an neighbourhood V of 0, the hypersurface $\{g = 0\}$ is reduced and $V \cap \{g = 0\} - \{0\}$ is a complex analytic manifold.

By using an analytic version of Corollary 2.8 of Milnor (1968), we obtain that the hypersurfaces $\{g = t\}$ are nonsingular in V when V is a sufficiently small neighbourhood of 0 and $t \neq 0$ is small enough.

We may suppose that V contains the closed ball $\mathbb{B}_\varepsilon(0)$. As J. Milnor observed, the boundary $\partial\mathbb{B}_\varepsilon(0) = \mathbb{S}_\varepsilon(0)$ intersects transversally the hypersurface $\{g = 0\}$ in \mathbb{C}^{n+1} . Therefore, by continuity of the transversality for t small enough, the hypersurface $\{g = t\}$ intersects transversally the sphere $\mathbb{S}_\varepsilon(0)$ in \mathbb{C}^{n+1} . So, there exists a number $\eta(\varepsilon)$ such that the hypersurface $\{g = t\}$ intersects $\mathbb{S}_\varepsilon(0)$ whenever $|t| \leq \eta(\varepsilon)$.

One can choose $\eta(\varepsilon)$ such that the intersections $\{g = t\} \cap \mathbb{B}_\varepsilon(0)$, when $0 < |t| \leq \eta(\varepsilon)$, is not singular.

One considers the manifold $\mathbb{B}_\varepsilon(0) \cap g^{-1}(\partial\mathbb{D}_{\eta(\varepsilon)})$ with boundary, where $\partial\mathbb{D}_{\eta(\varepsilon)}$ is the circle $\mathbb{S}_{\eta(\varepsilon)}$ centered at 0 with radius $\eta(\varepsilon)$.

Let $\varphi(\varepsilon, \eta(\varepsilon))$ be the map induced by g from $\mathbb{B}_\varepsilon \cap g^{-1}(\mathbb{S}_{\eta(\varepsilon)})$ to $\mathbb{S}_{\eta(\varepsilon)}$. Now, the smooth map $\varphi(\varepsilon, \eta(\varepsilon))$ has no critical point inside the interior $\mathbb{B}_\varepsilon(0) \cap g^{-1}(\mathbb{S}_{\eta(\varepsilon)})$ as the fibers over $\mathbb{S}_{\eta(\varepsilon)}$ induced by g are nonsingular and furthermore, the restriction to the boundary $\mathbb{S}_\varepsilon(0) \cap g^{-1}(\mathbb{S}_{\eta(\varepsilon)})$ does not have critical point because of the transversality of the fibers with $\mathbb{S}_\varepsilon(0)$ in \mathbb{C}^{n+1} .

The Ehresmann lemma (see e.g., Lemma 6.2.10 of Lê et al., 2020) implies that the map $\varphi(\varepsilon, \eta)$ is a locally trivial smooth fibration.

A natural question is to ask if the fibrations of Theorem 2.1 and Theorem 2.2 are the same or not.

The answer lies in the book of J. Milnor. One can prove using Theorem 5.11 of Milnor (1968) that the fibration φ_ε and the one induced by g on $\mathbb{B}_\varepsilon(0) \cap g^{-1}(\mathbb{S}_{\eta(\varepsilon)})$ are the same in the case where one knows that both fibrations are locally trivial smooth fibrations.

Remark 2.1. *Let $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ be a germ of a complex analytic map such that the fiber $F^{-1}(0)$ of F over 0 has dimension $n - p$ and has an isolated singularity at 0. One can show that there are representatives U of $(\mathbb{C}^n, 0)$ and V of $(\mathbb{C}^p, 0)$ such that F induces a map $F : U \rightarrow V$ that restricts to the space $U - \Sigma$ outside the critical space Σ of $F : U \rightarrow V$ that can be stratified with the Thom property. Using the observations that we do above, one can easily prove this fact.*

3. NONISOLATED SINGULARITIES

In Hamm and Lê (1973), the authors proved without knowing it that if the singularity at 0 of the hypersurface $g = 0$ is not isolated, one can stratify the analytic function g such that the restriction to $U - \{0\}$, where U is a small neighbourhood of 0 in \mathbb{C}^{n+1} , is stratified with the Thom condition.

Namely, in Hamm and Lê (1973), in the case of a complex analytic function, we call good stratification the Whitney stratification of the function that satisfies the Thom condition outside the point $\{0\}$. Following an idea of F. Pham, we give a proof of Theoreme 1.2.1 in Hamm and Lê (1973). In order to prove that theorem, according to the idea of F. Pham, we use the inequality of Łojasiewicz for the complex analytic function g near the point 0. There is an open neighbourhood Ω of 0 in \mathbb{C}^{n+1} such that there exists a number θ , $1 > \theta > 0$, for which, for any point z in Ω , we have:

$$\|\text{grad}g(z)\| \geq |g(z)|^\theta$$

where $\text{grad}g(z)$ is the complex gradient of g at z (see Milnor, 1968, p. 33), i.e., the

coordinates of the complex gradient are the conjugate of the partial derivatives of g :

$$\text{grad } g(z) = (\overline{\partial g / \partial z_0(z)}, \dots, \overline{\partial g / \partial z_n(z)}),$$

and $\|\bullet\|$ is the Hermitian norm.

Then, one can choose an integer k such that:

$$k > \frac{1}{1 - \theta}.$$

Then, the hypersurface given by $\{g - T^k = 0\}$ in the space $\mathbb{C}^{n+1} \times \mathbb{C}$ is "transverse" to the space $\mathbb{C}^{n+1} \times \{0\}$ in the sense that there is an open neighbourhood of $(0, 0)$ such that the limits of the tangents at a sequence of nonsingular points of the hypersurface $\{g - T^k = 0\}$ tending to a point of $\{g = 0\} \times \{0\}$ is transverse to the hyperplane $\{T = 0\} = \mathbb{C}^{n+1} \times \{0\}$ (see Hamm and Lê, 1973, p. 324). We obtain a neighborhood W of 0, and a Whitney stratification of W in which $W - \{g = 0\}$ and $\{0\}$ are strata and $\{g = 0\} \cap W$ is a union of strata. So outside 0, that stratification of $W - \{0\}$ satisfies the Thom property relative to the function $g|_{W - \{0\}}$.

This observation implies that we have a fibration analogous to Theorem 2.2. As we saw before, the fibers $\{g = t\}$ of g are nonsingular for $t \neq 0$ small enough, as a consequence of a statement similar to Corollary 2.8 of Milnor (1968) in the case of an complex analytic function. Namely, we have:

Proposition 2.1. Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a complex analytic function at the point 0 of \mathbb{C}^{n+1} . Then, there is a representative $f : U \rightarrow V$ of f , such that the representative has only a finite number of distinct critical values.

Consider an open neighbourhood of $\{g = 0\} \cap U$. We may choose:

$$(|g| < \alpha) \cap U,$$

where α is small enough so that the restriction of g to $(|g| < \alpha) \cap U$ has only the critical value 0. Choose ε small enough such that $\mathbb{B}_\varepsilon(0)$ is contained in $(|g| < \alpha) \cap U$, such that the strata with the Thom property that intersect $\mathbb{B}_\varepsilon(0)$ have dimension ≥ 1 and intersect $\mathbb{S}_\varepsilon(0)$ transversally in \mathbb{C}^{n+1} .

Now, let z_n be a sequence of points of $\mathbb{S}_\varepsilon(0) - \{g = 0\}$ that tends to a point z of $\mathbb{S}_\varepsilon \cap \{g = 0\}$. Because of the Thom property, by choosing properly the sequence (z_n) , the sequence of tangent spaces $T_{z_n}(g^{-1}(g(z_n)))$ has a limit \mathbb{T} that contains the tangent at z to the stratum S_z , which contains z . Since S_z intersects the sphere $\mathbb{S}_\varepsilon(0)$ transversally in \mathbb{C}^{n+1} , the spaces $T_{z_n}(g^{-1}(g(z_n)))$ for n big enough are also transverse to the sphere $\mathbb{S}_\varepsilon(0)$ in \mathbb{C}^{n+1} . Thus, for α small enough, the fibers $\{g = t\}$ for $t, \alpha \geq t > 0$ intersect $\mathbb{S}_\varepsilon(0)$ transversally in \mathbb{C}^{n+1} .

Then, we can repeat the proof that we did in the case of an isolated singularity to obtain Theorem 2.2 in the case of a hypersurface with a nonisolated singularity.

4. GENERAL CASE

In fact, a fibration theorem as Theorem 2.2 is true for any germ of a complex analytic function of a complex reduced analytic space. We have:

Theorem 4.1. *Let $(X, 0)$ be the germ of a reduced complex analytic space. Let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a complex analytic function. We assume that $(X, 0)$ is embedded in $(\mathbb{C}^N, 0)$. Considering a representative $f : U \rightarrow V$ of $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ and $i : U \hookrightarrow U_0$ a representative of the embedding $i : (X, 0) \rightarrow (\mathbb{C}^N, 0)$, there is ε_0 small enough such that, for any ε , $\varepsilon_0 \geq \varepsilon > 0$, there is $\eta(\varepsilon) > 0$ for which:*

$$X \cap \mathbb{B}_\varepsilon(0) \cap f^{-1}(\partial \mathbb{D}_{\eta(\varepsilon)}(0)) \rightarrow \partial \mathbb{D}_{\eta(\varepsilon)}(0)$$

is a locally trivial topological fibration.

However, notice that instead of a locally trivial smooth fibration as in Theorem 2.2, we have a locally trivial \mathcal{C}^0 fibration.

Proof.

The proof uses two main ingredients: one is the existence of a stratified function with the Thom condition and the other one is the Thom-Mather first isotopy for Whitney stratifications. \square

Choose a representative $f : U \rightarrow V$ of the germ $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ where the neighborhoods U of 0 in X and V of 0 in \mathbb{C} are such that U has a Whitney stratification \mathcal{S} where the reduced space $|\{f = 0\}|$ is a union of strata. We may also suppose that U is small enough and $f : U \rightarrow V$ is surjective. We may also suppose that $V - \{0\}$ and that $\{0\}$ is the Whitney stratification of V .

If U is small enough, the only possible stratum of dimension 0 is the point $\{0\}$ and all the strata of U contain the point $\{0\}$ in their closure. Consider the Whitney stratification outside $\{0\}$, i.e., the Whitney stratification of $U - \{0\}$, and the stratification of V stratifying $f : U - \{0\} \rightarrow V$. This stratification of f satisfies the Thom condition because the Whitney stratification of $U - \{0\}$ is locally trivial, required in Theorem 4.2.1 of Briaçon et al. (1994, p. 541). Of course, it is supposed that $X \neq \{0\}$ or that f is not constant.

Now, the fibers $\{f = t\}$ for $t \neq 0$ small enough are transverse to the strata of \mathcal{S} outside $\{f = 0\}$ by using again Corollary 2.8 of Milnor (1968) adapted to the local analytic case.

So, on $\mathbb{B}_\varepsilon(0) \cap X \cap f^{-1}(\partial \mathbb{D}_{\eta(\varepsilon)}(0))$ we have the stratification induced by \mathcal{S} . As stated by Corollary 2.8 of Milnor (1968), for t small enough, the hypersurfaces $\{f = t\}$ intersect the strata of $U - \{f = 0\}$ transversally. Since there are only a finite number of strata to which the point 0 is adherent, we obtain that there is η_0 such the hypersurfaces $\{f = t\}$ for $\eta_0 \geq |t| > 0$ are transverse to all the strata of $U - \{f = 0\}$.

On the other hand, the Thom condition implies that there is η_1 such that if $\eta_1 \geq |t| > 0$, the hypersurface $\{f = t\}$ is transverse to $\mathbb{S}_\varepsilon(0) \cap S_\alpha$ for any stratum S_α of \mathcal{S} that intersects $\mathbb{S}_\varepsilon(0)$.

This implies that the restriction of f to the strata of the Whitney stratification induced by the Whitney stratification \mathcal{S} to $\mathbb{B}_\varepsilon(0) \cap X \cap f^{-1}(\partial\mathbb{D}_{\eta(\varepsilon)}(0))$, where we can choose $\eta(\varepsilon) = \inf(\eta_0, \eta_1)$, gives a stratified map:

$$X \cap \mathbb{B}_\varepsilon(0) \cap f^{-1}(\partial\mathbb{D}_{\eta(\varepsilon)}(0)) \rightarrow \partial\mathbb{D}_{\eta(\varepsilon)}(0)$$

that satisfies the Thom-Mather first isotopy theorem. This is why that map is a locally trivial \mathcal{C}^0 fibration.

Notice that the use of the Ehresmann lemma as in the proof of Theorem 2.2 is replaced in the general case by the Thom-Mather first isotopy theorem. However, the locally trivial fibration obtained by the Ehresmann lemma is smooth when the locally trivial fibration obtained using the Thom-Mather isotopy theorem is continuous.

We have used Theorem 4.2.1 of Briançon et al. (1994). We could have used Corollary 1 of Hironaka (1977, p. 248). However, in Corollary 1, Hironaka assumed that the map is proper. Therefore, we have the following:

Problem. Can one deduce Theorem 4.2.1 of Briançon et al. (1994) from Corollary 1 of Hironaka (1977, p. 248)? One may have to use Theorem 2 of Hironaka (1977, p. 247).

5. A QUESTION POSED BY DELIGNE

Around 1980, P. Deligne asked me in a letter if, for any map $f : X \rightarrow Y$, there exists a modification $\sigma : \tilde{Y} \rightarrow Y$ of Y such that the pull-back $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ of f by σ has vanishing cycles. I could not answer this question, but it came to my mind that there could exist a modification of his question such that the pull-back is a stratified map satisfying the Thom condition.

In fact, my student C. Sabbah solved the problem using results in Hironaka (1977).

Examples.

- (1) Consider the example given above of a stratified map that does not satisfy the Thom condition. We have the blowing-up of a point, say 0, in \mathbb{C}^2 . Then, we have a complex analytic map $e : \tilde{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$. We know that the map e can be stratified, but there is no stratification that satisfies the Thom condition. However, if one pulls back e by the modification e itself, then one obtains a map which can be stratified with the Thom condition.

(2) Let $F : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^2, 0)$ be defined by $F(x, y, z) = x^2 - y^2z$. As Sabbah noticed, this map cannot be stratified with the Thom condition. In this case, if one considers (u, v) to be coordinates of \mathbb{C}^2 , the blowing-up of the ideal (u, v^2) of \mathbb{C}^2 is a map $\sigma : Z \rightarrow \mathbb{C}^2$, and the pull-back of F by σ gives the map \tilde{F} that satisfies the Thom condition.

In fact, Sabbah proves the following theorems in Sabbah (1983, pp. 287-288):

Theorem 5.1. (C. Sabbah) *Let $f : X \rightarrow Y$ be a proper complex analytic map between reduced complex analytic spaces of finite dimension. Let \mathcal{S} be a stratification of f . There is a blowing-up $\pi : \tilde{Y} \rightarrow Y$ that is a sequence of blowing-ups such that the pull-back $\tilde{f} : X \times_Y \tilde{Y} \rightarrow \tilde{Y}$ of f by π is stratified by the stratification $\tilde{\mathcal{S}}$ that is compatible with the stratification \mathcal{S} and satisfies the Thom condition.*

In the second theorem, Sabbah considers nonproper maps. His statement is the following:

Theorem 5.2. (C. Sabbah) *Let $f : X \rightarrow Y$ be a complex analytic map between reduced spaces. Let y_0 be a point of Y and let L be a compact in the fiber $f^{-1}(y_0)$. Let \mathcal{S}_X (resp.) \mathcal{S}_Y be a complex analytic stratification of X (resp. Y). There is a finite number of blowing-ups $\pi_i : Y_i \rightarrow Y$ ($1 \leq i \leq m$) each of which is the composition of a finite number of local blowing-ups, complete over a neighbourhood U of y_0 , a neighbourhood V of L and, for every i , a complex analytic stratification of $f_i : V_i \rightarrow Y_i$ pull-back of the restriction $f|_V$ by π_i , compatible with \mathcal{S}_X and \mathcal{S}_Y such that (f_i, \mathcal{S}_i) satisfies the Thom condition,*

Notice that Sabbah used the terminology morphism without blowing-up that was used by R. Thom in his original paper (Thom, 1969) or good stratification for the stratification of a stratified map that satisfies the Thom condition which was the terminology used by H. Hamm and the author in their paper (Hamm & Lê, 1973). Here we make the abuse of language by saying that $g : X \rightarrow Y$ satisfies the Thom condition when there is a strict complex analytic subspace Z of X such that the restriction $g|_{X-Z} : X-Z \rightarrow Y$ satisfies the Thom condition.

Let $(X, 0) \rightarrow (Y, 0)$ be the germ of a surjective stratified map from an irreducible complex analytic germ $(X, 0)$ onto a connected germ $(Y, 0)$. If the stratification satisfies the Thom condition, one can define a general fiber whose homology, cohomology, and homotopy could define the vanishing cycles. The general fiber can be considered a geometric representation of the vanishing cycles.

So, Sabbah's theorem could be seen as a way to answer Deligne's question.

We set the following problem and conjecture:

Problem. Let $f : (X, 0) \rightarrow (Y, 0)$ be the germ of a surjective stratified map from an irre-

ducible complex analytic germ $(X, 0)$ onto a connected germ $(Y, 0)$. Suppose there exists a strict subspace Z of X such that the stratification restricted to $U - Z$ satisfies the Thom condition, where U is a sufficiently small neighbourhood of $\{0\}$ in X . Prove that one can define a general fiber of f at 0.

Conjecture. It is equivalent to have the existence of a general fiber for a complex analytic germ $(X, 0) \rightarrow (Y, 0)$, which is a surjective stratified map from an irreducible complex analytic germ $(X, 0)$ onto a connected germ $(Y, 0)$, and to ask for the existence of a strict subspace Z of X such that the restriction to $X - Z$ of a stratification with the Thom condition, where X is a sufficiently small representative of the germ $(X, 0)$.

Somehow, if this conjecture is true, then the Sabbah theorems are an answer to Deligne's question.

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